

# Global well-posedness and scattering for the defocusing, $L^2$ -critical, nonlinear Schrödinger equation when $d \geq 3$

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**Abstract:** In this paper we prove that the defocusing,  $d$ -dimensional mass critical nonlinear Schrödinger initial value problem is globally well-posed and scattering for  $u_0 \in L^2(\mathbf{R}^d)$  and  $d \geq 3$ . To do this, we will prove a frequency localized interaction Morawetz estimate similar to the estimate made in [10]. Since we are considering an  $L^2$  - critical initial value problem we will localize to low frequencies.

## 1 Introduction

The  $d$ -dimensional,  $L^2$  critical nonlinear Schrödinger initial value problem is given by

$$\begin{aligned} iu_t + \Delta u &= F(u), \\ u(0, x) &= u_0 \in L^2(\mathbf{R}^d), \end{aligned} \tag{1.1}$$

where  $F(u) = \mu|u|^{4/d}u$ ,  $\mu = \pm 1$ ,  $u(t) : \mathbf{R}^d \rightarrow \mathbf{C}$ . When  $\mu = +1$  (1.1) is said to be defocusing and when  $\mu = -1$  (1.1) is said to be focusing. The term  $L^2$  - critical refers to scaling. If  $u(t, x)$  solves (1.1) on  $[0, T]$  with initial data  $u(0, x) = u_0(x)$ , then

$$\lambda^{d/2}u(\lambda^2 t, \lambda x) \tag{1.2}$$

solves (1.1) on  $[0, \frac{T}{\lambda^2}]$  with initial data  $\lambda^{d/2}u_0(\lambda x)$ . The scaling preserves the  $L^2(\mathbf{R}^d)$  norm.

$$\|\lambda^{d/2}u_0(\lambda x)\|_{L_x^2(\mathbf{R}^d)} = \|u_0(x)\|_{L_x^2(\mathbf{R}^d)}. \tag{1.3}$$

It was observed in [4] that the solution to (1.1) conserves the quantities mass,

$$M(u(t)) = \int |u(t, x)|^2 dx = M(u(0)), \tag{1.4}$$

and energy

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{\mu d}{2(d+2)} \int |u(t, x)|^{\frac{2d+4}{d}} dx = E(u(0)). \quad (1.5)$$

**Remark:** When  $\mu = +1$  this quantity is positive definite.

A solution to (1.1) obeys Duhamel's formula.

**Definition 1.1**  $u : I \times \mathbf{R}^d \rightarrow \mathbf{C}$ ,  $I \subset \mathbf{R}$  is a solution to (1.1) if for any compact  $J \subset I$ ,  $u \in C_t^0 L_x^2(J \times \mathbf{R}^d) \cap L_{t,x}^{\frac{2(d+2)}{d}}(J \times \mathbf{R}^d)$ , and for all  $t, t_0 \in I$ ,

$$u(t) = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau. \quad (1.6)$$

The space  $L_{t,x}^{\frac{2(d+2)}{d}}(J \times \mathbf{R}^d)$  arises from the Strichartz estimates. This norm is also invariant under the scaling (1.2).

**Definition 1.2** A solution to (1.1) defined on  $I \subset \mathbf{R}$  blows up forward in time if there exists  $t_0 \in I$  such that

$$\int_{t_0}^{\sup(I)} \int |u(t, x)|^{\frac{2(d+2)}{d}} dx dt = \infty. \quad (1.7)$$

$u$  blows up backward in time if there exists  $t_0 \in I$  such that

$$\int_{\inf(I)}^{t_0} \int |u(t, x)|^{\frac{2(d+2)}{d}} dx dt = \infty. \quad (1.8)$$

**Definition 1.3** A solution  $u(t, x)$  to (1.1) is said to scatter forward in time if there exists  $u_+ \in L^2(\mathbf{R}^d)$  such that

$$\lim_{t \rightarrow \infty} \|e^{it\Delta} u_+ - u(t, x)\|_{L^2(\mathbf{R}^d)} = 0. \quad (1.9)$$

A solution is said to scatter backward in time if there exists  $u_- \in L^2(\mathbf{R}^d)$  such that

$$\lim_{t \rightarrow -\infty} \|e^{it\Delta} u_- - u(t, x)\|_{L^2(\mathbf{R}^d)} = 0. \quad (1.10)$$

**Theorem 1.1** For any  $d \geq 1$ , there exists  $\epsilon(d) > 0$  such that if  $\|u_0\|_{L^2(\mathbf{R}^d)} < \epsilon(d)$ , then (1.1) is globally well-posed and scatters both forward and backward in time.

*Proof:* See [4], [5].  $\square$

We will recall the proof of this theorem in §2. [4], [5] also proved (1.1) is locally well-posed for  $u_0 \in L^2_x(\mathbf{R}^d)$  on some interval  $[0, T]$ , where  $T(u_0)$  depends on the profile of the initial data, not just its size in  $L^2(\mathbf{R}^d)$ .

**Theorem 1.2** *Given  $u_0 \in L^2(\mathbf{R}^d)$  and  $t_0 \in \mathbf{R}$ , there exists a maximal lifespan solution  $u$  to (1.1) defined on  $I \subset \mathbf{R}$  with  $u(t_0) = u_0$ . Moreover,*

1.  *$I$  is an open neighborhood of  $t_0$ .*
2. *If  $\sup(I)$  or  $\inf(I)$  is finite, then  $u$  blows up in the corresponding time direction.*
3. *The map that takes initial data to the corresponding solution is uniformly continuous on compact time intervals for bounded sets of initial data.*
4. *If  $\sup(I) = \infty$  and  $u$  does not blow up forward in time, then  $u$  scatters forward to a free solution. If  $\inf(I) = -\infty$  and  $u$  does not blow up backward in time, then  $u$  scatters backward to a free solution.*

*Proof:* See [4], [5].  $\square$

In the focusing case there are known counterexamples to (1.1) globally well-posed and scattering for all  $u_0 \in L^2(\mathbf{R}^d)$ . In the defocusing case there are no known counterexamples to global well-posedness and scattering for  $u_0 \in L^2(\mathbf{R}^d)$  of arbitrary size. Therefore, it has been conjectured,

**Conjecture 1.3** *For  $d \geq 1$ , the defocusing, mass critical nonlinear Schrödinger initial value problem (1.1),  $\mu = +1$  is globally well-posed for  $u_0 \in L^2(\mathbf{R}^d)$  and all solutions scatter to a free solution as  $t \rightarrow \pm\infty$ .*

This conjecture has been affirmed in the radial case.

**Theorem 1.4** *When  $d = 2$ ,  $\mu = +1$ , (1.1) is globally well-posed and scattering for  $u_0 \in L^2(\mathbf{R}^2)$  radial.*

*Proof:* See [21].

**Theorem 1.5** *When  $d \geq 3$ ,  $\mu = +1$ , (1.1) is globally well-posed and scattering for  $u_0 \in L^2(\mathbf{R}^d)$  radial.*

*Proof:* See [31], [23].

In this paper we remove the radial condition for the case when  $d \geq 3$  and prove

**Theorem 1.6** *(1.1) is globally well-posed and scattering for  $u_0 \in L^2(\mathbf{R}^d)$ ,  $d \geq 3$ .*

**Remark:** [21] and [23] also proved global well-posedness and scattering for the focusing, mass-critical initial value problem

$$\begin{aligned} iu_t + \Delta u &= -|u|^{4/d}u, \\ u(0, x) &= u_0, \end{aligned} \tag{1.11}$$

with radial data and mass less than the mass of the ground state. Many of the tools used in this paper to prove global well-posedness and scattering when  $\mu = +1$  can also be applied to the focusing problem with mass below the mass of the ground state. So whenever possible we will prove theorems for  $\mu = \pm 1$ .

**Outline of the Proof.** We prove this theorem via the concentration compactness method, a modification of the induction on energy method. The induction on energy method was introduced in [3] to prove global well-posedness and scattering for the defocusing energy-critical initial value problem in  $\mathbf{R}^3$  for radial data. [3] proved that it sufficed to treat solutions to the energy critical problem that were localized in both space and frequency. See [10], [25], [37], and [29] for more work on the defocusing, energy critical initial value problem.

This induction on energy method lead the development of the concentration compactness method. This method uses a concentration compactness technique to isolate a minimal mass/energy blowup solution. [21] and [23] used concentration compactness to prove theorems 1.4 and 1.5. Since (1.1) is globally well-posed for small  $\|u_0\|_{L^2(\mathbf{R}^d)}$ , if (1.1) is not globally well-posed for all  $u_0 \in L^2(\mathbf{R}^d)$ , then there must be a minimum  $\|u_0\|_{L^2(\mathbf{R}^d)} = m_0$  where global well-posedness fails. [33] showed that for conjecture 1.3 to fail, there must exist a minimal mass blowup solution with a number of additional properties. We show that such a solution cannot occur, proving theorem 1.6. See [18], [19], [20] for more information on this method.

**Definition 1.4** *A set is precompact in  $L^2(\mathbf{R}^d)$  if it has compact closure in  $L^2(\mathbf{R}^d)$ .*

**Definition 1.5** *A solution  $u(t, x)$  is said to be almost periodic if there exists a group of symmetries  $G$  of the equation such that  $\{u(t)\}/G$  is a precompact set.*

**Theorem 1.7** *Suppose conjecture 1.3 fails. Then there exists a maximal lifespan solution  $u$  on  $I \subset \mathbf{R}$ ,  $u$  blows up both forward and backward in time, and  $u$  is almost periodic modulo the group  $G = (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d$  which consists of scaling symmetries, translational symmetries, and Galilean symmetries. That is, for any  $t \in I$ ,*

$$u(t, x) = \frac{1}{N(t)^{d/2}} e^{ix \cdot \xi(t)} Q_t\left(\frac{x - x(t)}{N(t)}\right), \tag{1.12}$$

where  $Q_t(x) \in K \subset L^2(\mathbf{R}^d)$ ,  $K$  is a precompact subset of  $L^2(\mathbf{R}^d)$ .

Additionally,  $[0, \infty) \subset I$ ,  $N(t) \leq 1$  on  $[0, \infty)$ ,  $N(0) = 1$ , and

$$\int_0^\infty \int |u(t, x)|^{\frac{2(d+2)}{d}} dx dt = \infty. \quad (1.13)$$

*Proof:* See [33] and section four of [31].  $\square$

**Remark:** This is also true of a minimal mass blowup solution to the focusing problem (1.11).

**Remark:** From the Arzela-Ascoli theorem, a set  $K \subset L^2(\mathbf{R}^d)$  is precompact if and only if there exists a compactness modulus function,  $C(\eta) < \infty$  for all  $\eta > 0$  such that

$$\int_{|x| \geq C(\eta)} |f(x)|^2 dx + \int_{|\xi| \geq C(\eta)} |\hat{f}(\xi)|^2 d\xi < \eta. \quad (1.14)$$

To verify conjecture 1.3 in the case  $d \geq 3$  it suffices to consider two scenarios separately,

$$\int_0^\infty N(t)^3 dt = \infty, \quad (1.15)$$

and

$$\int_0^\infty N(t)^3 dt < \infty. \quad (1.16)$$

The main new ingredient of this paper is to prove a long-time Strichartz estimate. The proof of this estimate relies on the bilinear Strichartz estimates and an induction on frequency argument.

**Theorem 1.8** *Suppose  $J \subset [0, \infty)$  is compact,  $d \geq 3$ ,  $u$  is a minimal mass blowup solution to (1.1) for  $\mu = \pm 1$ , and  $\int_J N(t)^3 dt = K$ . Then there exists a function  $\rho(N)$ ,  $\rho(N) \leq 1$ ,  $\lim_{N \rightarrow \infty} \rho(N) = 0$ , such that for  $N \leq K$ ,*

$$\|P_{|\xi - \xi(t)| > N} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \lesssim_{m_0, d} \rho(N) \left(\frac{K}{N}\right)^{1/2}. \quad (1.17)$$

To preclude the scenario  $\int_0^\infty N(t)^3 dt = \infty$  we will rely on a frequency localized interaction Morawetz estimate. (See [10] for such an estimate in the energy-critical case. [10] dealt with the energy-critical equation,  $u(t) \in H^1$ , and thus truncated to high frequencies). The interaction Morawetz estimates scale like  $\int_J N(t)^3 dt$ , and in fact are bounded below by some constant times  $\int_J N(t)^3 dt$ . Since we are truncating to low frequencies, our method is very similar to the almost Morawetz estimates that are often used in conjunction with the I-method. (See [1], [7], [8], [9], [11], [6], [15], [14], [12], and [13] for more information on the I-method.) The estimates (1.17) enable us to control the errors that arise from frequency truncation and prove

**Theorem 1.9** *If  $\int_J N(t)^3 dt = K$ , and  $C$  is a large constant, independent of  $K$ , then*

$$\int_J \int_{\mathbf{R}^d \times \mathbf{R}^d} (-\Delta \Delta |x - y|) |P_{\leq CK} u(t, x)|^2 |P_{\leq CK} u(t, y)|^2 dx dy dt \lesssim_{m_0, d} o(K). \quad (1.18)$$

This leads to a contradiction in the case when  $\int_0^\infty N(t)^3 dt = \infty$ .

To deal with the case when  $\int_0^\infty N(t)^3 dt < \infty$ , we use a method similar to the method used in [21], [32], and [23]. Such a minimal mass blowup solution must possess additional regularity in particular  $u(t) \in L_t^\infty \dot{H}_x^s([0, \infty) \times \mathbf{R}^d)$  for  $0 < s < 1 + 4/d$ . Since  $\int N(t)^3 dt < \infty$ ,  $N(t) \searrow 0$  as  $t \rightarrow \infty$ , this contradicts conservation of energy. We rely on theorem 1.8 to prove this additional regularity.

**Outline of the Paper:** In §2, we describe some harmonic analysis and properties of the linear Schrödinger equation that will be needed later in the paper. In particular we discuss Strichartz estimates. Global well-posedness and scattering for small mass will be an easy consequence of these estimates. We discuss the movement of  $\xi(t)$  and  $N(t)$  for a minimal mass blowup solution in this section. We also quote bilinear Strichartz estimates and the fractional chain rule.

In §3 we prove theorem 1.8. We use these estimates in §4 to obtain the frequency localized interaction Morawetz estimate and in §5 to obtain additional regularity.

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## 2 The linear Schrödinger equation

In this section we will introduce some of the tools that will be needed later in the paper.

### Linear Strichartz Estimates:

**Definition 2.1** *A pair  $(p, q)$  will be called an admissible pair for  $d \geq 3$  if  $\frac{2}{p} = d(\frac{1}{2} - \frac{1}{q})$ , and  $p \geq 2$ .*

**Theorem 2.1** *If  $u(t, x)$  solves the initial value problem*

$$\begin{aligned} iu_t + \Delta u &= F(t), \\ u(0, x) &= u_0, \end{aligned} \quad (2.1)$$

*on an interval  $I$ , then*

$$\|u\|_{L_t^p L_x^q(I \times \mathbf{R}^d)} \lesssim_{p, q, \tilde{p}, \tilde{q}, d} \|u_0\|_{L^2(\mathbf{R}^d)} + \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'}(I \times \mathbf{R}^d)}, \quad (2.2)$$

*for all admissible pairs  $(p, q)$ ,  $(\tilde{p}, \tilde{q})$ .  $\tilde{p}'$  denotes the Lebesgue dual of  $\tilde{p}$ .*

*Proof:* See [30] for the case when  $p > 2$ ,  $\tilde{p} > 2$ , and [17] for the proof when  $p = 2$ ,  $\tilde{p} = 2$ , or both. We will rely very heavily on the double endpoint case, or when both  $p = 2$  and  $\tilde{p} = 2$ .

We will also make heavy use of the bilinear Strichartz estimates throughout the paper.

**Lemma 2.2** *Suppose  $\hat{v}(t, \xi)$  is supported on  $|\xi - \xi_0| \leq M$  and  $\hat{u}(t, \xi)$  is supported on  $|\xi - \xi_0| > N$ ,  $M \ll N$ ,  $\xi_0 \in \mathbf{R}^d$ . Then, for the interval  $I = [a, b]$ ,  $d \geq 1$ ,*

$$\|uv\|_{L_{t,x}^2(I \times \mathbf{R}^d)} \lesssim \frac{M^{(d-1)/2}}{N^{1/2}} \|u\|_{S_*^0(I \times \mathbf{R}^d)} \|v\|_{S_*^0(I \times \mathbf{R}^d)}, \quad (2.3)$$

where

$$\|u\|_{S_*^0(I \times \mathbf{R}^d)} \equiv \|u(a)\|_{L^2(\mathbf{R}^d)} + \|(i\partial_t + \Delta)u\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I \times \mathbf{R}^d)}. \quad (2.4)$$

*Proof:* See [37].

We will also need the Littlewood-Paley partition of unity. Let  $\phi \in C_0^\infty(\mathbf{R}^d)$ , radial,  $0 \leq \phi \leq 1$ ,

$$\phi(x) = \begin{cases} 1, & |x| \leq 1; \\ 0, & |x| > 2. \end{cases} \quad (2.5)$$

Define the frequency truncation

$$\mathcal{F}(P_{\leq N}u) = \phi\left(\frac{\xi}{N}\right)\hat{u}(\xi). \quad (2.6)$$

Let  $P_{>N}u = u - P_{\leq N}u$  and  $P_Nu = P_{\leq 2N}u - P_{\leq N}u$ . For convenience of notation let  $u_N = P_Nu$ ,  $u_{\leq N} = P_{\leq N}u$ , and  $u_{>N} = P_{>N}u$ .

The Strichartz estimates motivate the definition of the Strichartz space.

**Definition 2.2** *Define the norm*

$$\|u\|_{S^0(I \times \mathbf{R}^d)} \equiv \sup_{(p,q) \text{ admissible}} \|u\|_{L_t^p L_x^q(I \times \mathbf{R}^d)}. \quad (2.7)$$

$$S^0(I \times \mathbf{R}^d) = \{u \in C_t^0(I, L^2(\mathbf{R}^d)) : \|u\|_{S^0(I \times \mathbf{R}^d)} < \infty\}. \quad (2.8)$$

We also define the space  $N^0(I \times \mathbf{R}^d)$  to be the space dual to  $S^0(I \times \mathbf{R}^d)$  with appropriate norm. Then in fact,

$$\|u\|_{S^0(I \times \mathbf{R}^d)} \lesssim \|u_0\|_{L^2(\mathbf{R}^d)} + \|F\|_{N^0(I \times \mathbf{R}^d)}. \quad (2.9)$$

**Theorem 2.3** (1.1) is globally well-posed when  $\|u_0\|_{L^2(\mathbf{R}^d)}$  is small.

*Proof:* By (2.9) and the definition of  $S^0$ ,  $N^0$ ,

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}((-\infty,\infty)\times\mathbf{R}^d)} \lesssim_d \|u_0\|_{L^2(\mathbf{R}^d)} + \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}((-\infty,\infty)\times\mathbf{R}^d)}^{1+4/d}. \quad (2.10)$$

By the continuity method, if  $\|u_0\|_{L^2(\mathbf{R}^d)}$  is sufficiently small, then we have global well-posedness. We can also obtain scattering with this argument.  $\square$

Now let

$$A(m) = \sup\{\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}((-\infty,\infty)\times\mathbf{R}^d)} : u \text{ solves (1.1), } \|u(0)\|_{L^2(\mathbf{R}^d)} = m\}. \quad (2.11)$$

If we can prove  $A(m) < \infty$  for any  $m$ , then we have proved global well-posedness and scattering. Indeed, partition  $(-\infty, \infty)$  into a finite number of subintervals with  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_j \times \mathbf{R}^d)} \leq \epsilon$  for each subinterval and iterate the argument in the proof of theorem 2.3.

Using a stability lemma from [33] we can prove that  $A(m)$  is a continuous function of  $m$ , which proves that  $\{m : A(m) = \infty\}$  is a closed set. This implies that if global well-posedness and scattering does not hold in the defocusing case for all  $u_0 \in L^2(\mathbf{R}^d)$ , then there must be a minimum  $m_0$  with  $A(m_0) = \infty$ . Furthermore, [33] proved that for conjecture 1.3 to fail, there must exist a maximal interval  $I \subset \mathbf{R}$  with  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)} = \infty$ , and  $u$  blows up both forward and backward in time. Moreover, this minimal mass blowup solution must be concentrated in both space and frequency. For any  $\eta > 0$ , there exists  $C(\eta) < \infty$  with

$$\int_{|x-x(t)| \geq \frac{C(\eta)}{N(t)}} |u(t, x)|^2 dx < \eta, \quad (2.12)$$

and

$$\int_{|\xi-\xi(t)| \geq C(\eta)N(t)} |\hat{u}(t, \xi)|^2 d\xi < \eta. \quad (2.13)$$

By the Arzela-Ascoli theorem this proves  $\{u(t, x)\}/G$  is a precompact. It is quite clear that shifting the origin generates a  $d$ -dimensional symmetry group for solutions to (1.1), and by (1.2) changing  $N(t)$  by a fixed constant also generates the multiplicative symmetry group  $(0, \infty)$  for solutions to (1.1). The Galilean transformation generates the  $d$ -dimensional phase shift symmetry group.

**Theorem 2.4** Suppose  $u(t, x)$  solves



$$\begin{aligned} iu_t + \Delta u &= F(u), \\ u(0, x) &= u_0. \end{aligned} \tag{2.14}$$

Then  $v(t, x) = e^{-it|\xi_0|^2} e^{ix \cdot \xi_0} u(t, x - 2\xi_0 t)$  solves the initial value problem

$$\begin{aligned} iv_t + \Delta v &= F(v), \\ v(0, x) &= e^{ix \cdot \xi_0} u(0, x). \end{aligned} \tag{2.15}$$

*Proof:* This follows by direct calculation.  $\square$

If  $u(t, x)$  obeys (2.12) and (2.13) and  $v(t, x) = e^{-it|\xi_0|^2} e^{ix \cdot \xi_0} u(t, x - 2\xi_0 t)$ , then

$$\int_{|\xi - \xi_0 - \xi(t)| \geq C(\eta)N(t)} |\hat{v}(t, \xi)|^2 d\xi < \eta, \tag{2.16}$$

$$\int_{|x - 2\xi_0 t - x(t)| \geq \frac{C(\eta)}{N(t)}} |v(t, x)|^2 dx < \eta. \tag{2.17}$$

**Remark:** This will be useful to us later because it shifts  $\xi(t)$  by a fixed amount  $\xi_0 \in \mathbf{R}^d$ . For example, this allows us to set  $\xi(0) = 0$ . We now need to obtain some information on the movement of  $N(t)$  and  $\xi(t)$ .

**Lemma 2.5** *If  $J$  is an interval with*

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J \times \mathbf{R}^d)} \leq C, \tag{2.18}$$

*then for  $t_1, t_2 \in J$ ,*

$$N(t_1) \sim_{C, m_0} N(t_2). \tag{2.19}$$

*Proof:* See [21], corollary 3.6.  $\square$

**Lemma 2.6** *If  $u(t, x)$  is a minimal mass blowup solution on an interval  $J$ ,*

$$\int_J N(t)^2 dt \lesssim \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J \times \mathbf{R}^d)}^{\frac{2(d+2)}{d}} \lesssim 1 + \int_J N(t)^2 dt. \tag{2.20}$$

*Proof:* See [23].

**Lemma 2.7** Suppose  $u$  is a minimal mass blowup solution with  $N(t) \leq 1$ . Suppose also that  $J$  is some interval partitioned into subintervals  $J_k$  with  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} = \epsilon$  on each  $J_k$ . Again let

$$N(J_k) = \sup_{J_k} N(t). \quad (2.21)$$

Then,

$$\sum_{J_k} N(J_k) \sim \int_J N(t)^3 dt. \quad (2.22)$$

*Proof:* Since  $N(t_1) \sim N(t_2)$  for  $t_1, t_2 \in J_k$  it suffices to show  $|J_k| \sim \frac{1}{N(J_k)^2}$ . By Holder's inequality and (2.12),

$$\left(\frac{m_0}{2}\right)^{\frac{2(d+2)}{d}} \leq \left(\int_{|x-x(t)| \leq \frac{C(\frac{m_0^2}{1000})}{N(t)}} |u(t, x)|^2 dx\right)^{\frac{d+2}{d}} \lesssim_{m_0} \frac{1}{N(t)^2} \|u(t, x)\|_{L_x^{\frac{2(d+2)}{d}}(\mathbf{R}^d)}^{\frac{2(d+2)}{d}}.$$

Therefore,

$$\int_{J_k} N(t)^2 dt \lesssim_{m_0} \epsilon,$$

so  $|J_k| \lesssim \frac{1}{N(J_k)^2}$ . Moreover, by Duhamel's formula, if  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} = \epsilon$  then

$$\|e^{i(t-a_k)\Delta} u(a_k)\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} \geq \frac{\epsilon}{2},$$

where  $J_k = [a_k, b_k]$ . By Sobolev embedding,

$$\|e^{i(t-a_k)\Delta} P_{|\xi-\xi(a_k)| \leq C(\epsilon^2)N(a_k)} u(a_k)\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} \lesssim_{m_0} N(J_k)^2 |J_k|. \quad (2.23)$$

Therefore,  $|J_k| \gtrsim \frac{1}{N(J_k)^2}$ . Summing up over subintervals proves the lemma.  $\square$

We can use this fact to control the movement of  $\xi(t)$ . This control is essential for the arguments in the paper.

**Lemma 2.8** Partition  $J = [0, T_0]$  into subintervals  $J = \cup J_k$  such that

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} \leq \epsilon, \quad (2.24)$$

where  $\epsilon$  is the same  $\epsilon$  as in lemma 2.5. Let  $N(J_k) = \sup_{t \in J_k} N(t)$ . Then

$$|\xi(0) - \xi(T_0)| \lesssim \sum_k N(J_k), \quad (2.25)$$

which is the sum over the intervals  $J_k$ .

*Proof:* See lemma 5.18 of [22].  $\square$

Possibly after adjusting the modulus function  $C(\eta)$  in (2.12), (2.13) by a constant, we can choose  $\xi(t) : I \rightarrow \mathbf{R}^d$  such that

$$|\frac{d}{dt}\xi(t)| \lesssim_d N(t)^3. \quad (2.26)$$

**Fractional Chain Rule:** Another essential tool that we will need is a good analysis of embedding Holder continuous functions into Sobolev spaces. Since  $d \geq 3$  our analysis of (1.1) will be complicated by the fact that the nonlinearity  $F(u) = \mu|u|^{4/d}u$  is no longer algebraic. Because of this fact, the Fourier transform of  $F(u)$  is not the convolution of Fourier transforms of  $u$ , and thus  $F(P_{<N})$  need not be truncated in frequency. Instead, we will use the fractional chain rule.

**Lemma 2.9** *Let  $G$  be a Holder continuous function of order  $0 < \alpha < 1$ . Then for every  $0 < s < \alpha$ ,  $1 < p < \infty$ ,  $\frac{s}{\alpha} < \sigma < 1$ ,*

$$\| |\nabla|^s G(u) \|_{L_x^p(\mathbf{R}^d)} \lesssim \| |u|^{\alpha - \frac{s}{\sigma}} \|_{L_x^{p_1}(\mathbf{R}^d)} \| |\nabla|^\sigma u \|_{L_x^{\frac{s}{\sigma} p_2}(\mathbf{R}^d)}^{s/\sigma}. \quad (2.27)$$

*Proof:* See [37].

**Corollary 2.10** *Let  $0 \leq s < 1 + 4/d$ . Then on any spacetime slab  $I \times \mathbf{R}^d$ ,*

$$\| |\nabla|^s F(u) \|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I \times \mathbf{R}^d)} \lesssim \| |\nabla|^s u \|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)} \| u \|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)}^{4/d}. \quad (2.28)$$

*Proof:* See [23].

**Corollary 2.11** *For  $0 \leq s < 1 + \frac{4}{d}$ ,*

$$\| |\nabla|^s F(u) \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)} \lesssim \| u \|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)}^{4/d} \| |\nabla|^s u \|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}. \quad (2.29)$$

*Proof:* We use an argument similar to the argument found in [23] to prove corollary 2.10. The case  $s \leq 1$  follows from  $\nabla F(u) = O(|u|^{4/d})(\nabla u)$  and interpolating with the estimate for  $\mu|u|^{4/d}u$ . Now consider  $s > 1$ .

**Case 1:**,  $d = 4$

$$\begin{aligned} & \|\Delta F(u)\|_{L_t^2 L_x^{4/3}(J \times \mathbf{R}^4)} \\ &= \|F_z(u)\Delta u + F_{\bar{z}}(u)\Delta \bar{u} + F_{zz}(u)(\nabla u)^2 + F_{\bar{z}\bar{z}}(\nabla \bar{u})^2 + 2F_{z\bar{z}}(u)|\nabla u|^2\|_{L_t^2 L_x^{4/3}(J \times \mathbf{R}^4)}. \end{aligned} \quad (2.30)$$

By interpolation

$$\|\nabla u\|_{L_t^4 L_x^{8/3}(J \times \mathbf{R}^4)}^2 \lesssim \|\Delta u\|_{L_t^2 L_x^4(J \times \mathbf{R}^4)} \|u\|_{L_t^\infty L_x^2(J \times \mathbf{R}^4)},$$

which proves the corollary in this case.

**Case 2:**  $d > 4$ : Use the chain rule and fractional product rule (see [34] for more details).

$$\begin{aligned} & \| |\nabla|^s F(u) \|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \lesssim \|F_z(u) + F_{\bar{z}}(u)\|_{L_t^\infty L_x^{d/2}(J \times \mathbf{R}^d)} \| |\nabla|^s u \|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \\ & + \| |\nabla|^{s-1} [F_z(u) + F_{\bar{z}}(u)] \|_{L_t^{\frac{2s}{s-1}} L_x^q(J \times \mathbf{R}^d)} \|\nabla u\|_{L_t^{2s} L_x^p(J \times \mathbf{R}^d)}, \end{aligned} \quad (2.31)$$

with

$$\frac{1}{p} = \frac{(d-2)}{2ds} + \frac{s-1}{2s}, \quad (2.32)$$

$$\frac{1}{q} = \frac{2}{d} + \frac{(s-1)(d-2)}{2ds} - \frac{s-1}{2s}. \quad (2.33)$$

By interpolation,

$$\|\nabla u\|_{L_t^{2s} L_x^p(J \times \mathbf{R}^d)} \lesssim \| |\nabla|^s u \|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}^{1/s} \|u\|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)}^{(s-1)/s}. \quad (2.34)$$

Now use lemma 2.13. Choose  $\sigma$  with  $\frac{s-1}{4/d} < \sigma < 1$ . Let  $\frac{1}{p_1} = \frac{2}{d} - \frac{s-1}{2\sigma}$  and  $\frac{1}{p_2} = \frac{(s-1)(d-2)}{2ds} + \frac{(s-\sigma)(s-1)}{2s\sigma}$ . Both  $F_z(z)$  and  $F_{\bar{z}}(z)$  are Holder continuous functions of order  $\frac{4}{d}$ . Without loss of generality consider  $F_z(u)$ .

$$\| |\nabla|^{s-1} F_z(u(t)) \|_{L_x^q(\mathbf{R}^d)} \lesssim |u(t)|^{4/d - \frac{s-1}{\sigma}} \| |\nabla|^\sigma u(t) \|_{L_x^{\frac{s-1}{\sigma} p_2}(\mathbf{R}^d)}^{\frac{s-1}{\sigma}}. \quad (2.35)$$

By interpolation

$$\| |\nabla|^\sigma u(t) \|_{L_t^{\frac{2s}{\sigma}} L_x^{(\frac{s-1}{s})p_2}(J \times \mathbf{R}^d)}^{\frac{s-1}{\sigma}} \lesssim \| |\nabla|^s u \|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}^{\frac{s-1}{s}} \| u \|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)}^{(\frac{s-1}{\sigma})(\frac{s-\sigma}{s})}. \quad (2.36)$$

Finally,

$$\| |u|^{4/d - \frac{s-1}{\sigma}} \|_{L_t^\infty L_x^{p_1}(J \times \mathbf{R}^d)} \lesssim \| u \|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)}^{2/p_1}. \quad (2.37)$$

Summing up our terms, the corollary is proved in this case also.

**Case 3,  $d = 3$ :** Take  $2 \leq s < 7/3$ .

$$\begin{aligned} & \| |\nabla|^s F(u) \|_{L_t^2 L_x^{6/5}(J \times \mathbf{R}^3)} \\ &= \| |\nabla|^{s-2} [F_z(u) \Delta u + F_{\bar{z}}(u) \Delta \bar{u} + 2F_{z\bar{z}}(u) |\nabla u|^2 + F_{zz}(u) (\nabla u)^2 + F_{\bar{z}\bar{z}}(u) (\nabla \bar{u})^2] \|_{L_t^2 L_x^{6/5}(J \times \mathbf{R}^3)}. \end{aligned} \quad (2.38)$$

$F_{zz}, F_{z\bar{z}}, F_{\bar{z}\bar{z}}$  are Holder continuous of order  $1/3$ , while  $F_z$  and  $F_{\bar{z}}$  are in fact differentiable, so use lemma 2.13 and interpolate as in the previous case.  $\square$

Finally, at various points in the proof of theorem 1.6 we will also rely on the Sobolev embedding lemma.

**Lemma 2.12** *If  $\frac{1}{p} = \frac{1}{2} - \frac{\rho}{d}$  and  $\rho < \frac{d}{2}$ , then*

$$\dot{H}^\rho(\mathbf{R}^d) \subset L^p(\mathbf{R}^d),$$

and

$$\| u \|_{L^p(\mathbf{R}^d)} \lesssim_{p,d} \| u \|_{\dot{H}^\rho(\mathbf{R}^d)}.$$

We will also rely on the Hardy-Littlewood-Sobolev lemma.

**Lemma 2.13** *Suppose  $\frac{r}{d} = 1 - (\frac{1}{p} - \frac{1}{q})$ ,  $1 < p < \infty$ ,  $1 < q < \infty$ , and  $0 < r < d$ . Then let*

$$G(x) = \int \frac{1}{|x-y|^r} F(y) dy. \quad (2.39)$$

$$\| G \|_{L^q(\mathbf{R}^d)} \lesssim \| F \|_{L^p(\mathbf{R}^d)}. \quad (2.40)$$

*We will use this result in §4 a great deal.*

### 3 Long-time Strichartz Estimates

In order to defeat the minimal mass blowup solution we will obtain Strichartz estimates over long time intervals. These estimates will be used in §4 to preclude scenario (1.15) from occurring and in §5 to preclude scenario (1.16).

**Theorem 3.1** *Suppose  $u$  is a minimal mass blowup solution to (1.1),  $\mu = \pm 1$ ,  $J$  is a compact interval with  $N(t) \leq 1$ , and*

$$\int_J N(t)^3 dt = K. \quad (3.1)$$

*Then for  $N \leq K$ , there exists a constant  $C_3(m_0, d)$  such that*

$$\|P_{|\xi - \xi(t)| > N} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \leq C_3(m_0, d) \frac{K^{1/2}}{N^{1/2}}. \quad (3.2)$$

*Proof:* We prove this theorem by induction on  $N$ . Start with the base case.

**Lemma 3.2** *Since  $J$  is compact and  $N(t) \leq 1$ ,*

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J \times \mathbf{R}^d)} = C(J) < \infty.$$

*Therefore, theorem 3.1 is true for  $N \leq \frac{K}{C(J)}$ .*

*Proof:* Partition  $J$  into  $\frac{C^{2+4/d}}{\epsilon^{2+4/d}}$  subintervals  $J_k$  with  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} = \epsilon$ . By Duhamel's formula and Strichartz estimates,

$$\|u\|_{S^0(J_k \times \mathbf{R}^d)} \lesssim_d \|u_0\|_{L^2(\mathbf{R}^d)} + \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)}^{1+4/d} \lesssim_{m_0, d} 1, \quad (3.3)$$

which implies

$$\|u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \leq C_1(m_0, d) C(J)^{1/2}. \quad (3.4)$$

This implies theorem 2.1 is true for the interval  $J$  when  $N \leq \frac{K}{C(J)}$ .

Next, we will make the inductive step. In the interest of first exposing the main idea, we will obtain an estimate conducive to induction when  $\xi(t) \equiv 0$ . After this, we will treat the case when  $\xi(t)$  is time dependent, which necessarily introduces a few additional complications.

**Remark:** The case  $\xi(t) \equiv 0$  is already fairly interesting on its own. It includes the radial case, but also includes the case when  $u(0, x)$  is symmetric across the  $x_1, \dots, x_d$  axes.

**Lemma 3.3** *If  $\xi(t) \equiv 0$ , then there exists a function  $\delta(C_0)$ ,  $\delta(C_0) \rightarrow 0$  as  $C_0 \rightarrow \infty$ , such that when  $d = 3$ ,*

$$\begin{aligned} \|u_{>N}\|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} &\lesssim_{m_0, d, s} \|u_{>N}\|_{L_t^\infty L_x^2(J \times \mathbf{R}^3)} + \sum_{M \leq \eta N} \left(\frac{M}{N}\right)^s \|u_{>M}\|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} \\ &+ \delta(C_0) \|u_{>\eta N}\|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} + \frac{C_0^{3/2} K^{1/2}}{(\eta N)^{1/2}} \left(\sup_{J_k} \|u_{>\eta N}\|_{S_*^0(J_k \times \mathbf{R}^3)}\right). \end{aligned} \quad (3.5)$$

When  $d \geq 4$ ,

$$\begin{aligned} \|u_{>N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} &\lesssim_{m_0, d, s} \|u_{>N}\|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)} + \sum_{M \leq \eta N} \left(\frac{M}{N}\right)^s \|u_{>M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \\ &+ \delta(C_0) \|u_{>\eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} + \frac{C_0^{4-6/d} K^{2/d}}{(\eta N)^{2/d}} \left(\sup_{J_k} \|u_{>\eta N}\|_{S_*^0(J_k \times \mathbf{R}^d)}\right)^{4/d} \|u_{>\eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}^{1-4/d}. \end{aligned} \quad (3.6)$$

*Proof:* Define a cutoff  $\chi(t) \in C_0^\infty(\mathbf{R}^d)$  in physical space,

$$\chi(t, x) = \begin{cases} 1, & |x - x(t)| \leq \frac{C_0}{N(t)}; \\ 0, & |x - x(t)| > \frac{2C_0}{N(t)}. \end{cases} \quad (3.7)$$

$C_0$  will be specified later.

$$\begin{aligned} \|P_{>N}(|u(\tau)|^{4/d} u(\tau))\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)} &\lesssim_d \|P_{>N}(|u_{\leq \eta N}|^{4/d} u_{\leq \eta N})\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)} \\ &+ \|(u_{>\eta N})|u_{>C_0 N(t)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)} + \|(u_{>\eta N})|(1 - \chi(t))u_{\leq C_0 N(t)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)} \\ &+ \|(u_{>\eta N})|\chi(t)u_{\leq C_0 N(t)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)}. \end{aligned} \quad (3.8)$$

By Bernstein's inequality and (2.29), for any  $0 \leq s < 1 + 4/d$ ,

$$\|P_{>N}(|u_{\leq \eta N}|^{4/d} u_{\leq \eta N})\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)} \lesssim_d \frac{1}{N^s} \|\nabla|^s u_{\leq \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \|u\|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)}^{4/d} \quad (3.9)$$

$$\lesssim_{m_0, d} \sum_{M \leq \eta N} \left(\frac{M}{N}\right)^s \|u_{>M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}. \quad (3.10)$$

For the next two terms we use (2.12) and (2.13). Since mass is concentrated in both frequency and space, we can deal with the mass outside these balls perturbatively.

$$\begin{aligned}
& \| (u_{>\eta N}) |u_{\leq C_0 N(t)}|^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)} + \| (u_{>\eta N}) |1 - \chi(t)| u_{\leq C_0 N(t)}|^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)} \\
& \leq \| u_{>\eta N} \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)} [ \| (1 - \chi(t)) u \|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)}^{4/d} + \| u_{\leq C_0 N(t)} \|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)}^{4/d} ] \\
& \leq \delta(C_0) \| u_{>\eta N} \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)},
\end{aligned}$$

with  $\delta(C_0) \rightarrow 0$  as  $C_0 \rightarrow \infty$  (see (2.12), (2.13)). Finally, take

$$\| (P_{>\eta N} u) | \chi(t) u_{\leq C_0 N(t)} |^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)}. \quad (3.11)$$

We will use (2.3) to estimate (3.11) on each subinterval  $J_k$  and then sum over all the subintervals.

$$\| u \|_{S_*^0(J_k \times \mathbf{R}^d)} = \| u_0 \|_{L^2(\mathbf{R}^d)} + \| |u|^{4/d} u \|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J_k \times \mathbf{R}^d)} \lesssim_{m_0, d} 1. \quad (3.12)$$

**When  $d = 3$ :** Recall that  $N(J_k) = \sup_{t \in J_k} N(t)$ . Applying the bilinear estimates, mass conservation  $\| u \|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)} = m_0$ , and Holder's inequality,

$$\begin{aligned}
& \| (P_{>\eta N} u) | \chi(t) u_{\leq C_0 N(t)} |^{4/3} \|_{L_t^2 L_x^{6/5}(J_k \times \mathbf{R}^3)} \\
& \leq \| (P_{>\eta N} u) (u_{\leq C_0 N(J_k)}) \|_{L_{t,x}^2(J_k \times \mathbf{R}^3)} \| \chi(t) \|_{L_t^\infty L_x^6(J_k \times \mathbf{R}^d)} \| u \|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)}^{1/3} \\
& \lesssim_{m_0, d} \frac{C_0 N(J_k)}{(\eta N)^{1/2}} \left( \frac{C_0}{N(J_k)} \right)^{1/2} \| u_{>\eta N} \|_{S_*^0(J_k \times \mathbf{R}^d)} \| u \|_{S_*^0(J_k \times \mathbf{R}^d)}.
\end{aligned}$$

Summing over the subintervals  $J_k$  and using lemma 2.7,

$$\| (P_{>\eta N} u) | \chi u_{\leq C_0 N(t)} |^{4/3} \|_{L_t^2 L_x^{6/5}(J \times \mathbf{R}^3)} \lesssim_{m_0, d} \frac{C_0^{3/2}}{\eta^{1/2}} \frac{K^{1/2}}{N^{1/2}} \left( \sup_{J_k} \| u_{>\eta N} \|_{S_*^0(J_k \times \mathbf{R}^d)} \right).$$

**When  $d \geq 4$ :**

To simplify notation let  $\frac{1}{q} = \frac{2(d-2)}{d^2}$  and  $\frac{1}{p} = \frac{1}{q} + \frac{2}{d}$ .

$$\| (P_{>\eta N} u) | \chi(t) u_{\leq C_0 N(t)} |^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)}$$



$$\leq \| |(P_{>\eta N} u)(u_{\leq C_0 N(t)})|^{4/d} (\chi(t))^{4/d} \|_{L_t^{d/2} L_x^p(J \times \mathbf{R}^d)} \| (P_{>\eta N} u)^{1-4/d} \|_{L_t^{2d/(d-4)} L_x^{\frac{2d^2}{(d-2)(d-4)}}(J \times \mathbf{R}^d)}.$$

Now,

$$\begin{aligned} & \| [(P_{>\eta N} u)(u_{\leq C_0 N(t)})]^{4/d} (\chi(t))^{4/d} \|_{L_t^{d/2} L_x^p(J_k \times \mathbf{R}^d)} \\ & \leq \| (P_{>\eta N} u)(u_{\leq C_0 N(t)}) \|_{L_{t,x}^2(J_k \times \mathbf{R}^d)}^{4/d} \| (\chi(t))^{4/d} \|_{L_t^\infty L_x^q(J_k \times \mathbf{R}^d)} \\ & \lesssim_d \frac{(C_0 N(J_k))^{\frac{2(d-1)}{d}}}{(\eta N)^{2/d}} \| u_{>\eta N} \|_{S_*^0(J_k \times \mathbf{R}^d)}^{4/d} \| u \|_{S_*^0(J_k \times \mathbf{R}^d)}^{4/d} \left( \frac{C_0}{N(J_k)} \right)^{\frac{2(d-2)}{d}} \\ & \lesssim_{m_0,d} C_0^{4-6/d} \left( \frac{N(J_k)}{\eta N} \right)^{2/d} \| u_{>\eta N} \|_{S_*^0(J_k \times \mathbf{R}^d)}^{4/d}. \end{aligned}$$

Again summing over all subintervals,

$$\begin{aligned} & \| [(P_{>\eta N} u)(u_{\leq C_0 N(t)})]^{4/d} (\chi(t))^{4/d} \|_{L_t^{d/2} L_x^p(J \times \mathbf{R}^d)} \\ & \lesssim_{m_0,d} \left( \sum N(J_k) \right)^{2/d} \frac{C_0^{4-6/d}}{(\eta N)^{2/d}} \left( \sup_{J_k} \| u_{>\eta N} \|_{S_*^0(J_k \times \mathbf{R}^d)} \right)^{4/d} \\ & \lesssim_{m_0,d} \frac{K^{2/d} C_0^{4-6/d}}{N^{2/d} \eta^{2/d}} \left( \sup_{J_k} \| u_{>\eta N} \|_{S_*^0(J_k \times \mathbf{R}^d)} \right)^{4/d}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \| (u_{>\eta N}) | \chi(t) u_{\leq C_0 N(t)} |^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)} \\ & \lesssim_{m_0,d} \frac{C_0^{2-4/d} K^{2/d}}{\eta^{2/d} N^{2/d}} \left( \sup_{J_k} \| u_{>\eta N} \|_{S_*^0(J_k \times \mathbf{R}^d)} \right)^{4/d} \| u_{>\eta N} \|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}^{1-4/d}. \end{aligned}$$

By Strichartz estimates, when  $d = 3$ ,

$$\begin{aligned} \| u_{>N} \|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} & \lesssim_{m_0,d,s} \| u_{>N} \|_{L_t^\infty L^2(J \times \mathbf{R}^3)} + \sum_{M \leq \eta N} \left( \frac{M}{N} \right)^s \| u_{>M} \|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} \\ & + \delta(C_0) \| u_{>\eta N} \|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} + \frac{C_0^{3/2} K^{1/2}}{(\eta N)^{1/2}} \left( \sup_{J_k} \| u_{>\eta N} \|_{S_*^0(J_k \times \mathbf{R}^3)} \right) \end{aligned} \tag{3.13}$$

This proves lemma 3.3 when  $d = 3$ . When  $d \geq 4$ ,

$$\begin{aligned} \|u_{>N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} &\lesssim_{m_0, d, s} \|u_{>N}\|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)} + \sum_{M \leq \eta N} \left(\frac{M}{N}\right)^s \|u_{>M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \\ &+ \delta(C_0) \|u_{>\eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} + \frac{C_0^{4-6/d} K^{2/d}}{(\eta N)^{2/d}} \left(\sup_{J_k} \|u_{>\eta N}\|_{S_*^0(J_k \times \mathbf{R}^d)}\right)^{4/d} \|u_{>\eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}^{1-4/d}. \end{aligned} \quad (3.14)$$

This proves lemma 3.3.  $\square$

Formulas (3.13) and (3.14) are quite good enough for us to prove theorem 3.1 by induction, as will be shown in a moment. When  $\xi(t)$  is time dependent we will settle for a slightly more complicated estimate.

**$\xi(t)$  time dependent:** When  $\xi(t)$  is time dependent we run into a bit of difficulty with the projection of the Duhamel term. Consider the case when  $J = [0, T]$ ,  $d = 3$ ,  $N(t) \equiv 1$  and  $\xi(t) = (t, 0, 0)$  to illustrate this idea. The low frequencies at time  $t = 0$  will be the high frequencies at some later time. Indeed, at time  $t > N$ ,  $\xi = 0$  will belong to the set

$$\{|\xi - \xi(t)| > N\}.$$

Therefore, we cannot use the exact same argument as in the case when  $\xi(t) \equiv 0$  because the projection

$$\|P_{|\xi - \xi(t)| > N}(|u|^{4/3}(u))\|_{L_t^2 L_x^{6/5}([0, T] \times \mathbf{R}^n)}$$

cannot be controlled by

$$\|P_{|\xi - \xi(t)| > \eta N} u\|_{L_t^2 L_x^6([0, T] \times \mathbf{R}^n)}.$$

Instead, we will partition  $J$  into subintervals where  $|\xi(t_1) - \xi(t_2)| \lesssim N$  on each of the subintervals and use the Duhamel formula on each subinterval separately. By lemma 2.8,

$$|\xi(a) - \xi(b)| \lesssim_d \int_a^b N(t)^3 dt.$$

So if  $\int_a^b N(t)^3 dt \ll N$ , we can use the Duhamel formula and the triangle inequality to say

$$\|P_{|\xi-\xi(t)|>N}u\|_{L_t^2 L_x^6([a,b]\times\mathbf{R}^3)} \lesssim_d \|P_{|\xi-\xi(a)|>\frac{N}{2}}u(a)\|_{L_x^2(\mathbf{R}^3)} + \|P_{|\xi-\xi(a)|>\frac{N}{2}}(|u|^{4/3}u)\|_{L_t^2 L_x^{6/5}(J\times\mathbf{R}^3)} \quad (3.15)$$

$$\lesssim_d \|P_{|\xi-\xi(a)|>\frac{N}{2}}u(a)\|_{L_x^2(\mathbf{R}^3)} + \|P_{|\xi-\xi(\tau)|>\frac{N}{4}}(|u|^{4/3}u)(\tau)\|_{L_t^2 L_x^{6/5}(J\times\mathbf{R}^3)}. \quad (3.16)$$

The tradeoff is that we are required to compute  $\|P_{|\xi-\xi(t)|>N}u\|_{L_t^2 L_x^6}$  over a bunch of subsets of  $J$  separately and then add up their  $L_t^2 L_x^6$  norms.

**Lemma 3.4** *Suppose  $\xi(t)$  is time dependent, and  $u$  satisfies the same conditions as theorem 3.1.*

$$\|u_{|\xi-\xi(t)|\geq N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J\times\mathbf{R}^d)} \lesssim_{m_0,d,s} \left(\frac{K}{N} + 1\right)^{1/2} \|u_{|\xi-\xi(t)|\geq \frac{N}{2}}\|_{L_t^\infty L_x^2(J\times\mathbf{R}^d)} + (\#B_j)^{1/2} \quad (3.17)$$

$$+ \sum_{M\leq\eta N} \left(\frac{M}{N}\right)^s \|u_{|\xi-\xi(t)|\geq M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J\times\mathbf{R}^d)} + \delta(C_0) \|u_{|\xi-\xi(t)|\geq \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J\times\mathbf{R}^d)} \quad (3.18)$$

$$+ \begin{cases} C_0^{3/2} \left(\frac{K}{\eta N}\right)^{1/2} (\sup_{J_k} \|u_{|\xi-\xi(t)|\geq \eta N}\|_{S_*^0(J_k\times\mathbf{R}^d)}), & \text{if } d=3; \\ C_0^{4-6/d} \left(\frac{K}{\eta N}\right)^{2/d} \|u_{|\xi-\xi(t)|\geq \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J\times\mathbf{R}^d)}^{1-4/d} (\sup_{J_k} \|u\|_{S_*^0(J_k\times\mathbf{R}^d)})^{4/d}, & \text{if } d\geq 4. \end{cases} \quad (3.19)$$

$$+ \left(\frac{K}{\eta N}\right)^{1/2} \begin{cases} \|u_{|\xi-\xi(t)|\geq \eta N}\|_{L_t^\infty L_x^2(J\times\mathbf{R}^3)}, & \text{if } d=3; \\ \|u_{|\xi-\xi(t)|\geq \eta N}\|_{L_t^\infty L_x^2(J\times\mathbf{R}^3)}^{4/d}, & \text{if } d\geq 4. \end{cases} \quad (3.20)$$

$(\#B_j)$  is the number of subintervals  $J_k$  with  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k\times\mathbf{R}^d)} = \epsilon$  and  $N(J_k) > \frac{\eta_1 N}{2}$ . As in the case when  $\xi(t) \equiv 0$ ,  $\delta(C_0) \rightarrow 0$  as  $C_0 \rightarrow \infty$ .

*Proof:* By lemma 2.8 we can choose  $\eta_1(d)$  sufficiently small so that  $|\xi(t_1) - \xi(t_2)| \leq \frac{N(J_k)}{100\eta_1(d)}$  for  $t_1, t_2 \in J_k$ . Since  $J$  is compact and  $N(t) \leq 1$ ,  $J$  is the union of a finite number of subintervals  $J_k$  with  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k\times\mathbf{R}^d)} = \epsilon$ . We will call these subintervals with  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k\times\mathbf{R}^d)} = \epsilon$  the  $\epsilon$ -subintervals.

We will call the  $\epsilon$ -subintervals with  $N(J_k) > \frac{\eta_1 N}{2}$  the bad subintervals. Then we will rewrite  $J = \cup G_j \cup B_j$ , where  $B_j$  are the bad  $\epsilon$ -subintervals and  $G_j$  are the collections of good  $\epsilon$ -subintervals in between the bad subintervals. Because  $\sum N(J_k) \sim_d K$ ,

$$(\#B_j) \lesssim_d \frac{2K}{N\eta_1}.$$

Next, cut each  $G_j$  into some subcollections of  $\epsilon$  - subintervals  $G_j = \cup_l G_{j,l}$  with

$$\sum N(J_k) \leq \eta_1 N \quad (3.21)$$

on each  $G_{j,l}$ , and such that one of three things is true about each  $G_{j,l}$ :

1.

$$\frac{\eta_1 N}{2} \leq \sum_{J_k: J_k \cap G_{j,l} \neq \emptyset} N(J_k) \leq \eta_1 N, \quad (3.22)$$

2.  $G_{j,l}$  is adjacent to  $B_{j+1}$ ,

or

3.  $G_{j,l}$  is at the end of  $J$ .

It is always possible to do this, because if  $G_{j,l}$  is not adjacent to  $B_{j+1}$  or the end of  $J$ , and

$$\sum_{J_k: J_k \cap G_{j,l} \neq \emptyset} N(J_k) < \frac{\eta_1 N}{2},$$

we can add the  $\epsilon$  - subinterval adjacent to  $G_{j,l}$  to  $G_{j,l}$  and still have

$$\sum_{J_k: G_{j,l}} N(J_k) \leq \eta_1 N.$$

Therefore,

$$(\#G_{j,l}) \lesssim_d (\#B_j) + 1 + \frac{2K}{N\eta_1}. \quad (3.23)$$

For the interval  $B_j$  we will be content to simply say

$$\|u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(B_j \times \mathbf{R}^d)} \lesssim 1 + \|u\|_{S^0(B_j \times \mathbf{R}^d)}^{1+4/d} \lesssim_{m_0, d} 1. \quad (3.24)$$

Now take  $G_{j,l} = [a_{jl}, b_{jl}]$ . By (3.21),  $|\xi(a_{jl}) - \xi(t)| \leq \frac{N}{100}$  when  $t \in G_{j,l}$ . This will give us something that is pretty close to (3.13) and (3.14) on each individual  $G_{j,l}$ .

**Lemma 3.5** *For  $G_{j,l} = [a_{jl}, b_{jl}]$ ,*

$$\begin{aligned}
\|P_{|\xi-\xi(t)|>N}u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(G_{j,l}\times\mathbf{R}^d)} &\lesssim_{m_0,d,s} \|P_{|\xi-\xi(a_{jl})|>\frac{N}{2}}u(a_{jl})\|_{L_x^2(\mathbf{R}^d)} \\
&+ \delta(C_0)\|P_{|\xi-\xi(t)|>\eta N}u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(G_{j,l}\times\mathbf{R}^d)} \\
&+ \|(u_{|\xi-\xi(t)|>\eta N})|\chi(t)u_{|\xi-\xi(t)|\leq C_0 N(t)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(G_{j,l}\times\mathbf{R}^d)} \\
&+ \sum_{M\leq\eta N} \left(\frac{M}{N}\right)^s \|u_{|\xi-\xi(t)|>M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(G_{j,l}\times\mathbf{R}^d)}.
\end{aligned} \tag{3.25}$$

*Proof:* By Duhamel's formula the solution on  $G_{j,l}$  has the form

$$u(t, x) = e^{i(t-a_{jl})\Delta}u(a_{jl}) - i \int_{a_{jl}}^t e^{i(t-\tau)\Delta}|u(\tau)|^{4/d}u(\tau)d\tau. \tag{3.26}$$

Because  $|\xi(a_{jl}) - \xi(t)| \leq \frac{N}{100}$ ,

$$\|P_{|\xi-\xi(t)|>N}u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(G_{j,l}\times\mathbf{R}^d)} \leq \|P_{|\xi-\xi(a_{jl})|>\frac{N}{2}}u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(G_{j,l}\times\mathbf{R}^d)} \tag{3.27}$$

$$\lesssim_d \|P_{|\xi-\xi(a_{jl})|>\frac{N}{2}}u(a_{jl})\|_{L_x^2(\mathbf{R}^d)} + \|P_{|\xi-\xi(a_{jl})|>\frac{N}{2}}(|u|^{4/d}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(G_{j,l}\times\mathbf{R}^d)}. \tag{3.28}$$

Turning to the Duhamel term,

$$\|P_{|\xi-\xi(a_{jl})|>\frac{N}{2}}(|u|^{4/d}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(G_{j,l}\times\mathbf{R}^d)} \tag{3.29}$$

$$\lesssim_d \|P_{|\xi-\xi(a_{jl})|>\frac{N}{2}}(|u_{|\xi-\xi(t)|\leq\eta N}|^{4/d}u_{|\xi-\xi(t)|\leq\eta N})\|_{L_t^2 L_x^{\frac{2d}{d+2}}(G_{j,l}\times\mathbf{R}^d)} \tag{3.30}$$

$$+ \|(u_{|\xi-\xi(t)|>\eta N})|u_{|\xi-\xi(t)|>C_0 N(t)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(G_{j,l}\times\mathbf{R}^d)} \tag{3.31}$$

$$+ \|(u_{|\xi-\xi(t)|>\eta N})(1-\chi(t))u|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(G_{j,l}\times\mathbf{R}^d)} \tag{3.32}$$

$$+ \|(u_{|\xi-\xi(t)|>\eta N})|\chi(t)u_{|\xi-\xi(t)|\leq C_0 N(t)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(G_{j,l}\times\mathbf{R}^d)}. \tag{3.33}$$

By (2.12) and (2.13),

$$\begin{aligned}
& \| (u_{|\xi-\xi(t)| \geq \eta N}) | (1 - \chi(t)) u_{|\xi-\xi(t)| \leq C_0 N(t)} |^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}(G_{j,l} \times \mathbf{R}^d)} \\
& + \| (u_{|\xi-\xi(t)| \geq \eta N}) | u_{|\xi-\xi(t)| > C_0 N(t)} |^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}(G_{j,l} \times \mathbf{R}^d)} \\
& \leq \delta(C_0) \| u_{|\xi-\xi(t)| \geq \eta N} \|_{L_t^2 L_x^{\frac{2d}{d-2}}(G_{j,l} \times \mathbf{R}^d)}.
\end{aligned} \tag{3.34}$$

This takes care of (3.31) and (3.32). Next take (3.30).

$$\| P_{|\xi-\xi(a_{jl})| > \frac{N}{2}} (|u_{|\xi-\xi(t)| \leq \eta N}|^{4/d} u_{|\xi-\xi(t)| \leq \eta N}) \|_{L_t^2 L_x^{\frac{2d}{d+2}}(G_{j,l} \times \mathbf{R}^d)} \tag{3.35}$$

$$= \| P_{|\xi-\xi(a_{jl})+\xi(t)| > \frac{N}{2}} (e^{-ix \cdot \xi(t)} |u_{|\xi-\xi(t)| \leq \eta N}|^{4/d} u_{|\xi-\xi(t)| \leq \eta N}) \|_{L_t^2 L_x^{\frac{2d}{d+2}}(G_{j,l} \times \mathbf{R}^d)} \tag{3.36}$$

$$= \| P_{|\xi-\xi(a_{jl})+\xi(t)| > \frac{N}{2}} (|e^{-ix \cdot \xi(t)} u_{|\xi-\xi(t)| \leq \eta N}|^{4/d} (e^{-ix \cdot \xi(t)} u_{|\xi-\xi(t)| \leq \eta N})) \|_{L_t^2 L_x^{\frac{2d}{d+2}}(G_{j,l} \times \mathbf{R}^d)}. \tag{3.37}$$

Because  $|\xi(a_{jl}) - \xi(t)| \leq \frac{N}{100}$  on  $G_{j,l}$ ,

$$(3.37) \leq \| P_{|\xi| > \frac{N}{4}} (|e^{-ix \cdot \xi(t)} u_{|\xi-\xi(t)| \leq \eta N}|^{4/d} (e^{-ix \cdot \xi(t)} u_{|\xi-\xi(t)| \leq \eta N})) \|_{L_t^2 L_x^{\frac{2d}{d+2}}(G_{j,l} \times \mathbf{R}^d)}. \tag{3.38}$$

By Bernstein's inequality,

$$(3.38) \lesssim_d \frac{1}{N^s} \| |\nabla|^s (|e^{-ix \cdot \xi(t)} u_{|\xi-\xi(t)| \leq \eta N}|^{4/d} (e^{-ix \cdot \xi(t)} u_{|\xi-\xi(t)| \leq \eta N})) \|_{L_t^2 L_x^{\frac{2d}{d+2}}(G_{j,l} \times \mathbf{R}^d)}. \tag{3.39}$$

By corollary 2.11, for  $0 \leq s < 1 + 4/d$ ,

$$(3.39) \lesssim_{m_0, d, s} \frac{1}{N^s} \| |\nabla|^s (e^{-ix \cdot \xi(t)} u_{|\xi-\xi(t)| \leq \eta N}) \|_{L_t^2 L_x^{\frac{2d}{d-2}}(G_{j,l} \times \mathbf{R}^d)}, \tag{3.40}$$

$$\lesssim_{m_0, d, s} \sum_{M \leq \eta N} \left( \frac{M}{N} \right)^s \| u_{|\xi-\xi(t)| > M} \|_{L_t^2 L_x^{\frac{2d}{d-2}}(G_{j,l} \times \mathbf{R}^d)}. \tag{3.41}$$

This finishes the proof of lemma 3.5.  $\square$

Returning to the proof of lemma 3.4, summing the estimates (3.25) over all the  $G_{j,l}$  intervals, and using the crude estimate (3.24) on each  $B_j$ ,

$$\|u_{|\xi-\xi(t)|>N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \lesssim_{m_0,d,s} (\sharp G_{j,l})^{1/2} \|u_{|\xi-\xi(t)|>\frac{N}{2}}\|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)} + (\sharp B_j)^{1/2} \quad (3.42)$$

$$+ \delta(C_0) \|u_{|\xi-\xi(t)|>\eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \quad (3.43)$$

$$+ \|(u_{|\xi-\xi(t)|>\eta N})|\chi(t)u_{|\xi-\xi(t)|\leq C_0 N(t)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \quad (3.44)$$

$$+ \sum_{M \leq \eta N} \left(\frac{M}{N}\right)^s \|u_{|\xi-\xi(t)|>M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \quad (3.45)$$

This is almost in an acceptable form for our purposes. All that we have left to do is make a bilinear estimate of (3.44). Take one of the  $\epsilon$  - subintervals  $J_k = [a_k, b_k]$ .

Suppose  $d = 3$  and  $N(J_k) \leq \eta_1 \eta N$ . We have  $|\xi(t) - \xi(a_k)| \leq \frac{N(J_k)}{\eta_1}$  for all  $t \in J_k$ . In particular,

$$\{\xi : |\xi - \xi(t)| \leq C_0 N(J_k)\} \subset \{\xi : |\xi - \xi(a_k)| \leq (C_0 + \frac{1}{\eta_1(d)})N(J_k)\}$$

and

$$\{|\xi - \xi(t)| \geq \eta N\} \subset \{\xi : |\xi - \xi(a_k)| \geq \frac{\eta N}{2}\}.$$

Therefore,

$$\begin{aligned} & \| (u_{|\xi-\xi(t)| \geq \eta N}) |\chi(t)u_{|\xi-\xi(t)| \leq C_0 N(t)}|^{4/3} \|_{L_t^2 L_x^{6/5}(J_k \times \mathbf{R}^3)} \\ & \leq \| (u_{|\xi-\xi(t)| \geq \eta N}) (u_{|\xi-\xi(t)| \leq C_0 N(J_k)}) \|_{L_{t,x}^2(J_k \times \mathbf{R}^3)} \| \chi(t) \|_{L_t^\infty L_x^6(J_k \times \mathbf{R}^3)} \| u \|_{L_t^\infty L_x^2(J_k \times \mathbf{R}^3)}^{1/3} \\ & \lesssim_{m_0,d} C_0^{1/2} \| (u_{|\xi-\xi(a_k)| \geq \frac{\eta N}{2}}) (u_{|\xi-\xi(a_k)| \leq (C_0 + \frac{1}{\eta_1(d)})N(J_k)}) \|_{L_{t,x}^2(J_k \times \mathbf{R}^3)} \\ & \lesssim_{m_0,d} \frac{C_0^{3/2}}{\eta^{1/2}} \frac{N(J_k)^{1/2}}{N^{1/2}} (\|u_{|\xi-\xi(a_k)| \geq \eta N}\|_{S_*^0(J_k \times \mathbf{R}^3)}), \end{aligned} \quad (3.46)$$

**Remark:** We take it for granted that  $C_0$  is large, in particular  $\gg \frac{1}{\eta_1}$ .

If  $N(J_k) \geq \eta \eta_1 N$  we simply say that since  $\|u\|_{L_t^2 L_x^6(J_k \times \mathbf{R}^3)} \lesssim_{m_0,d} 1$  and  $\|u\|_{L_t^\infty L_x^2(J_k \times \mathbf{R}^d)} = m_0$ ,

$$\| (u_{|\xi-\xi(t)| \geq \eta N}) |\chi(t)u_{|\xi-\xi(t)| \leq C_0 N(t)}|^{4/3} \|_{L_t^2 L_x^{6/5}(J_k \times \mathbf{R}^3)} \lesssim_{m_0,d} \|u_{|\xi-\xi(t)| \geq \eta N}\|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)}. \quad (3.47)$$

Because  $\sum N(J_k) \sim_d K$  there are  $\lesssim_d \frac{K}{\eta\eta_1 N}$  intervals with  $N(J_k) \geq \eta\eta_1 N$ .

Now take  $d \geq 4$ . Let  $\frac{1}{q} = \frac{2(d-2)}{d^2}$  and  $\frac{1}{p} = \frac{1}{q} + \frac{2}{d}$ . If  $N(J_k) \leq \eta_1 \eta N$ ,

$$\begin{aligned}
& \| |(u_{|\xi-\xi(t)| \geq \eta N}) \chi(t) (u_{|\xi-\xi(t)| \leq C_0 N(t)}) |^{4/d} \|_{L_t^{d/2} L_x^p(J_k \times \mathbf{R}^d)} \\
& \leq \| (u_{|\xi-\xi(t)| \geq \eta N}) (u_{|\xi-\xi(t)| \leq C_0 N(t)}) \|_{L_{t,x}^2(J \times \mathbf{R}^d)}^{4/d} \| \chi(t) \|_{L_t^\infty L_x^q(J \times \mathbf{R}^d)} \\
& \leq \| (u_{|\xi-\xi(a_k)| \geq \frac{\eta N}{2}}) (u_{|\xi-\xi(a_k)| \leq (C_0 + \frac{1}{\eta_1}) N(J_k)}) \|_{L_{t,x}^2(J_k \times \mathbf{R}^d)}^{4/d} \| \chi(t) \|_{L_t^\infty L_x^q(J \times \mathbf{R}^d)} \\
& \lesssim_{m_0,d} \frac{C_0^{4-6/d}}{\eta^{2/d}} \frac{N(J_k)^{2/d}}{N^{2/d}} (\| u_{|\xi-\xi(a_k)| \geq \frac{\eta N}{2}} \|_{S_*^0(J_k \times \mathbf{R}^d)})^{4/d}.
\end{aligned} \tag{3.48}$$

If  $N(J_k) \geq \eta\eta_1 N$ ,

$$\| (u_{|\xi-\xi(t)| \geq \eta N}) | \chi(t) u_{|\xi-\xi(t)| \leq C_0 N(t)} |^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} \lesssim_{m_0,d} \| u_{|\xi-\xi(t)| \geq \eta N} \|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)}^{4/d}. \tag{3.49}$$

Once again there are  $\lesssim_d \frac{K}{\eta_1 \eta N}$  subintervals with  $N(J_k) \geq \eta\eta_1 N$ .

Therefore, if  $d = 3$ ,

$$\begin{aligned}
& \| |(u_{|\xi-\xi(t)| \geq \eta N}) | \chi(t) u_{|\xi-\xi(t)| \leq C_0 N(t)} |^{4/3} \|_{L_t^2 L_x^{6/5}(J \times \mathbf{R}^3)} \\
& \lesssim_{m_0,d} \frac{K^{1/2} C_0^{3/2}}{(\eta N)^{1/2}} \left( \sup_{J_k; N(J_k) \leq \eta_1 \eta N} \| u_{|\xi-\xi(a_k)| \geq \frac{\eta N}{2}} \|_{S_*^0(J_k \times \mathbf{R}^3)} + \frac{K^{1/2}}{(\eta N)^{1/2}} \| u_{|\xi-\xi(t)| \geq \eta N} \|_{L_t^\infty L_x^2(J \times \mathbf{R}^3)} \right).
\end{aligned} \tag{3.50}$$

If  $d \geq 4$ ,

$$\begin{aligned}
& \| |(u_{|\xi-\xi(t)| \geq \eta N}) | \chi(t) u_{|\xi-\xi(t)| \leq C_0 N(t)} |^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)} \\
& \lesssim_{m_0,d} \frac{K^{2/d} C_0^{4-6/d}}{(\eta N)^{2/d}} \left( \sup_{J_k; N(J_k) \leq \eta_1 \eta N} \| u_{|\xi-\xi(a_k)| \geq \frac{\eta N}{2}} \|_{S_*^0(J_k \times \mathbf{R}^d)}^{4/d} \| u_{|\xi-\xi(t)| \geq \eta N} \|_{L_t^\infty L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}^{1-4/d} \right. \\
& \quad \left. + \frac{K^{1/2}}{(\eta N)^{1/2}} \| u_{|\xi-\xi(t)| \geq \eta N} \|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)}^{4/d} \right).
\end{aligned} \tag{3.51}$$

Summing up (3.42) - (3.45) and substituting (3.50) or (3.51) for (3.44), depending on dimension,

$$\| u_{|\xi-\xi(t)| \geq N} \|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \lesssim_{m_0,d,s} \left( \frac{K}{N} + 1 \right)^{1/2} \| u_{|\xi-\xi(t)| \geq \frac{N}{2}} \|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)} + (\sharp B_j)^{1/2} \tag{3.52}$$



$$+ \sum_{M \leq \eta N} \left(\frac{M}{N}\right)^s \|u_{|\xi-\xi(t)| \geq M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} + \delta(C_0) \|u_{|\xi-\xi(t)| \geq \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \quad (3.53)$$

$$+ \begin{cases} C_0^{3/2} \left(\frac{K}{\eta N}\right)^{1/2} (\sup_{J_k} \|u_{|\xi-\xi(a_k)| \geq \frac{\eta N}{2}}\|_{S_*^0(J_k \times \mathbf{R}^d)}), & \text{if } d = 3; \\ C_0^{4-6/d} \left(\frac{K}{\eta N}\right)^{2/d} \|u_{|\xi-\xi(a_k)| \geq \frac{\eta N}{2}}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}^{1-4/d} (\sup_{J_k} \|u\|_{S_*^0(J_k \times \mathbf{R}^d)})^{4/d}, & \text{if } d \geq 4. \end{cases} \quad (3.54)$$

$$+ \left(\frac{K}{\eta N}\right)^{1/2} \begin{cases} \|u_{|\xi-\xi(t)| \geq \eta N}\|_{L_t^\infty L_x^2(J \times \mathbf{R}^3)}, & \text{if } d = 3; \\ \|u_{|\xi-\xi(t)| \geq \eta N}\|_{L_t^\infty L_x^2(J \times \mathbf{R}^3)}^{4/d}, & \text{if } d \geq 4. \end{cases} \quad (3.55)$$

We have used  $(\sharp G_{j,l}) \lesssim_d \sharp(B_j) + 1 + (\frac{2K}{\eta N})$  and  $\sharp B_j \lesssim_d \frac{2K}{\eta N}$  in (3.52). The proof of lemma 3.4 is now complete.  $\square$

Now we are ready to prove theorem 3.1. Let  $s = 1$ . For now make the crude estimates  $\|u\|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)} \lesssim_{m_0} 1$  and

$$\sup_{J_k} \|u_{|\xi-\xi(a_k)| \geq \frac{\eta N}{2}}\|_{S_*^0(J_k \times \mathbf{R}^d)} \leq \sup_{J_k} \|u\|_{S_*^0(J_k \times \mathbf{R}^d)} \lesssim_{m_0, d} 1.$$

By (3.52) - (3.55),

$$\|u_{|\xi-\xi(t)| > N}\|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} \leq C_2(m_0, d) \left(\frac{K}{N}\right)^{1/2} + C_2(m_0, d) C_0^{3/2} \left(\frac{K}{\eta N}\right)^{1/2} \quad (3.56)$$

$$+ C_2(m_0, d) \sum_{M \leq \eta N} \left(\frac{M}{N}\right) \|u_{|\xi-\xi(t)| > M}\|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} + C_2(m_0, d) \delta(C_0) \|u_{|\xi-\xi(t)| > \eta N}\|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} \quad (3.57)$$

We can prove theorem 3.1 for  $d = 3$  by induction. Suppose theorem 3.1 is true for  $M \leq \eta N$ .

$$C_2(m_0, d) \sum_{M \leq \eta N} \left(\frac{M}{N}\right) \|u_{|\xi-\xi(t)| > M}\|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} \leq 5\eta^{1/2} C_2(m_0, d) C_3(m_0, d) \left(\frac{K}{N}\right)^{1/2}.$$

Choose  $\eta(m_0, d)$  sufficiently small so that  $\eta^{1/2} C_2(m_0, d) \leq \frac{1}{1000}$ .

Next,

$$\delta(C_0) C_2(m_0, d) \|u_{|\xi-\xi(t)| > \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \leq \delta(C_0) C_2(m_0, d) C_3(m_0, d) \left(\frac{K}{\eta N}\right)^{1/2}.$$

Since  $\delta(C_0) \rightarrow 0$  as  $C_0 \rightarrow \infty$ , choose  $C_0(\eta(m_0, d), m_0, d)$  sufficiently large so that  $\delta(C_0) \frac{C_2(m_0, d)}{\eta^{1/2}} \leq \frac{1}{1000}$ .

Finally, choose  $C_3(m_0, d)$  sufficiently large so that

$$C_2(m_0, d) + C_2(m_0, d) \frac{C_0(\eta(m_0, d), m_0, d)^{3/2}}{\eta(m_0, d)^{1/2}} \leq \frac{1}{1000} C_3(m_0, d).$$

This closes the induction and proves theorem 3.1 when  $d = 3$ .

We make a similar argument for  $d \geq 4$ .

$$\begin{aligned} \|u_{|\xi-\xi(t)|>N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} &\leq C_2(m_0, d) \left(\frac{K}{N}\right)^{1/2} + C_2(m_0, d) C_0^{4-6/d} \left(\frac{K}{\eta N}\right)^{2/d} \|u_{|\xi-\xi(t)|>\eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}^{1-4/d} \\ &\quad + C_2(m_0, d) \sum_{M \leq \eta N} \left(\frac{M}{N}\right) \|u_{|\xi-\xi(t)|>M}\|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} + C_2(m_0, d) \delta(C_0) \|u_{|\xi-\xi(t)|>\eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^3)} \end{aligned} \quad (3.58)$$

$$(3.59)$$

Choose  $\eta(m_0, d) > 0$  sufficiently small so that  $\eta^{1/2} C_2(m_0, d) \leq \frac{1}{1000}$ . Next, choose  $C_0(\eta(m_0, d), m_0, d)$  sufficiently large so that  $\delta(C_0) \frac{C_2(m_0, d)}{\eta^{1/2}} \leq \frac{1}{1000}$ . Finally, choose  $C_3(m_0, d)$  sufficiently large so that

$$C_2(m_0, d) + C_2(m_0, d) \frac{C_0^{4-6/d}}{\eta^{1/2}} \leq \frac{1}{1000} C_3(m_0, d)^{4/d}.$$

This closes the induction and proves theorem 3.1 when  $d \geq 4$ .  $\square$

For the upcoming section we will need

$$\|u_{|\xi-\xi(t)|>N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}$$

to decay slightly faster than  $(\frac{K}{N})^{1/2}$ .

**Theorem 3.6** *There exists a function  $\rho(N) \leq 1$ ,*

$$\lim_{N \rightarrow \infty} \rho(N) = 0, \quad (3.60)$$

*such that if  $u$  is a minimal mass blowup solution to (1.1),  $\mu = \pm 1$  on the compact interval  $J$  with  $N(t) \leq 1$  and  $\int_J N(t)^3 dt = K$ , then*

$$\|u_{|\xi-\xi(t)|>N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \leq C_3(m_0, d) \rho(N) \left(\frac{K}{N}\right)^{1/2}. \quad (3.61)$$

*Proof:* We will modify the argument of the proof of theorem 3.1 slightly, taking advantage of the decay afforded by (2.13),

$$\lim_{N \rightarrow \infty} \|u|_{\xi - \xi(t) > N}\|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)} = 0. \quad (3.62)$$

**Lemma 3.7** *Let  $J_k$  be an interval with  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} = \epsilon$ ,  $N(J_k) \leq 1$ , and let  $u$ ,  $J$  satisfy the hypotheses of theorem 3.6. Then there exists a function  $\sigma(N)$ ,  $\sigma(N) \lesssim_{m_0, d} 1$ ,  $\lim_{N \rightarrow \infty} \sigma(N) = 0$ , such that*

$$\sup_{J_k = [a_k, b_k] \subset J} \|u|_{\xi - \xi(a_k) > N}\|_{S_*^0(J_k \times \mathbf{R}^d)} \leq \sigma(N). \quad (3.63)$$

*Proof:* Since  $N(J_k) \leq 1$ ,  $|\xi(t) - \xi(a_k)| \leq \frac{1}{100\eta_1(d)}$  on  $J_k$ . Take  $N \geq \frac{1000}{\eta_1(d)}$ . The lemma follows from Strichartz estimates for  $N \leq \frac{1000}{\eta_1(d)}$ .

$$\begin{aligned} \|u|_{\xi - \xi(a_k) > N}\|_{S_*^0(J_k \times \mathbf{R}^d)} &\leq \|P|_{\xi - \xi(t) > \frac{N}{2}} u\|_{L_t^\infty L_x^2(J_k \times \mathbf{R}^d)} \\ &\quad + \|P|_{\xi - \xi(t) > \frac{N}{2}} (|u|^{4/d} u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J_k \times \mathbf{R}^d)}. \end{aligned} \quad (3.64)$$

By Bernstein's inequality,  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} \leq \epsilon$ ,

$$\begin{aligned} &\|P|_{\xi - \xi(t) \geq \frac{N}{2}} (|u|^{4/d} u|_{\xi - \xi(t) \leq \frac{N^{1/2}}{2}})\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J_k \times \mathbf{R}^d)} \\ &\lesssim \frac{1}{N} \|\nabla e^{-ix \cdot \xi(t)} (|u|^{4/d} u|_{\xi - \xi(t) \leq \frac{N^{1/2}}{2}})\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J_k \times \mathbf{R}^d)} \\ &\lesssim_d \frac{1}{N} \|\nabla(e^{-ix \cdot \xi(t)} u|_{\xi - \xi(t) \leq \frac{N^{1/2}}{2}})\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} \lesssim_{m_0, d} N^{-1/2}. \end{aligned} \quad (3.65)$$

Also,

$$\| |u|^{4/d} \|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J_k \times \mathbf{R}^d)} \leq \|u|_{\xi - \xi(t) \geq \frac{N^{1/2}}{2}}\|_{L_t^\infty L_x^2(J_k \times \mathbf{R}^d)}^{2/d} \|u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J_k \times \mathbf{R}^d)}^{1+2/d}. \quad (3.66)$$

Both (3.65) and (3.66) decay to 0 as  $N \nearrow \infty$ .  $\square$

Let

$$C_0(N) = \sup((\sup_{J_k \subset J} \|u|_{\xi - \xi(a_k) \geq N^{1/2}}\|_{S_*^0(J_k \times \mathbf{R}^d)})^{-1/100d}, \frac{1000}{\eta_1(d)}), \quad (3.67)$$

$$\eta(N) = \sup(\delta(C_0(N))^{-1/100}, C_0^{1/100}, 2N^{-1/2}). \quad (3.68)$$

By lemma 3.7,  $C_0(N) \nearrow \infty$ , which implies  $\eta(N) \searrow 0$ . By lemma 3.4,

$$\|u_{|\xi-\xi(t)| \geq N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \leq C_2(m_0, d) \left(\frac{K}{N} + 1\right)^{1/2} \|u_{|\xi-\xi(t)| \geq \frac{N}{2}}\|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)} \quad (3.69)$$

$$+ C_2(m_0, d) \sum_{M \leq \eta(N)N} \left(\frac{M}{N}\right) \|u_{|\xi-\xi(t)| > M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \quad (3.70)$$

$$+ \delta(C_0) \|u_{|\xi-\xi(t)| > \eta(N)N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \quad (3.71)$$

$$+ \begin{cases} C_2(m_0, d) C_0^{3/2} \left(\frac{K}{\eta N}\right)^{1/2} (\sup_{J_k} \|u_{|\xi-\xi(a_k)| \geq \frac{\eta(N)N}{2}}\|_{S_*^0(J_k \times \mathbf{R}^d)}), & \text{if } d = 3; \\ C_2(m_0, d) C_0^{4-6/d} \left(\frac{K}{\eta N}\right)^{2/d} \|u_{|\xi-\xi(t)| \geq \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}^{1-4/d} (\sup_{J_k} \|u\|_{S_*^0(J_k \times \mathbf{R}^d)})^{4/d}, & \text{if } d \geq 4. \end{cases} \quad (3.72)$$

$$+ C_2(m_0, d) \left(\frac{K}{\eta N}\right)^{1/2} \begin{cases} \|u_{|\xi-\xi(t)| \geq \eta N}\|_{L_t^\infty L_x^2(J \times \mathbf{R}^3)}, & \text{if } d = 3; \\ \|u_{|\xi-\xi(t)| \geq \eta N}\|_{L_t^\infty L_x^2(J \times \mathbf{R}^3)}^{4/d}, & \text{if } d \geq 4. \end{cases} \quad (3.73)$$

By theorem 3.1,

$$(3.70) \leq 5C_2(m_0, d) C_3(m_0, d) \eta(N)^{1/2} \left(\frac{K}{N}\right)^{1/2}.$$

$$(3.71) \leq C_2(m_0, d) C_3(m_0, d) \frac{\delta(C_0(N))}{\eta(N)^{1/2}} \left(\frac{K}{N}\right)^{1/2}.$$

When  $d = 3$ ,

$$(3.72) \leq C_2(m_0, d) \frac{C_0(N)^{3/2}}{\eta(N)^{1/2}} \left(\frac{K}{N}\right)^{1/2} (\sup_{J_k} \|u_{|\xi-\xi(a_k)| \geq \frac{\eta(N)N}{2}}\|_{S_*^0(J_k \times \mathbf{R}^d)}),$$

and when  $d \geq 4$ ,

$$(3.72) \leq C_2(m_0, d) C_3(m_0, d)^{1-4/d} \frac{C_0(N)^{4-6/d}}{\eta(N)^{1/2}} \left(\frac{K}{N}\right)^{1/2} (\sup_{J_k} \|u_{|\xi-\xi(a_k)| \geq \frac{\eta(N)N}{2}}\|_{S_*^0(J_k \times \mathbf{R}^d)})^{4/d}.$$

When  $d = 3$ , let

$$\begin{aligned}\tilde{\rho}(N) &= \frac{C_2(m_0, d)}{C_3(m_0, d)} \|u_{|\xi-\xi(t)| \geq \frac{N}{2}}\|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)} + C_2(m_0, d)(\eta(N))^{1/2} + C_2(m_0, d) \frac{\delta(C_0(N))}{\eta(N)^{1/2}} \\ &\quad + \frac{C_2(m_0, d)}{C_3(m_0, d)} \frac{C_0(N)^{3/2}}{\eta(N)^{1/2}} \left( \sup_{J_k} \|u_{|\xi-\xi(a_k)| \geq \frac{\eta(N)N}{2}}\|_{S_*^0(J_k \times \mathbf{R}^d)} \right),\end{aligned}\tag{3.74}$$

and when  $d \geq 4$  let

$$\begin{aligned}\tilde{\rho}(N) &= \frac{C_2(m_0, d)}{C_3(m_0, d)} \|u_{|\xi-\xi(t)| \geq \frac{N}{2}}\|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)} + C_2(m_0, d)(\eta(N))^{1/2} + C_2(m_0, d) \frac{\delta(C_0(N))}{\eta(N)^{1/2}} \\ &\quad + \frac{C_2(m_0, d)}{C_3(m_0, d)^{4/d}} \frac{C_0(N)^{6-4/d}}{\eta(N)^{1/2}} \left( \sup_{J_k} \|u_{|\xi-\xi(a_k)| \geq \frac{\eta(N)N}{2}}\|_{S_*^0(J_k \times \mathbf{R}^d)} \right)^{4/d}.\end{aligned}\tag{3.75}$$

This implies that for  $N \leq K$ ,

$$\|u_{|\xi-\xi(t)| > N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \leq C_3(m_0, d) \tilde{\rho}(N) \left(\frac{K}{N}\right)^{1/2}.\tag{3.76}$$

Lemma 3.7, (3.67), (3.68) imply  $\tilde{\rho}(N) \rightarrow 0$  as  $N \rightarrow \infty$ . Taking  $\rho(N) = \inf(1, \tilde{\rho}(N))$  proves the theorem.  $\square$

**Remark:** These estimates also hold for  $u$  a minimal mass blowup solution to the focusing initial value problem (1.11).

$$4 \quad \int_0^\infty N(t)^3 dt = \infty$$

We will defeat this scenario by proving a frequency localized interaction Morawetz estimate. The interaction Morawetz estimate was proved for solutions to the defocusing nonlinear Schrödinger equation in [8] when  $d = 3$ , and in [32] for dimensions  $d \geq 4$ . The interaction Morawetz estimate was proved by taking the tensor product of two solutions to (1.1). Let  $x$  refer to the first  $d$  variables in  $\mathbf{R}^d \times \mathbf{R}^d$  and  $y$  refer to the second  $d$  variables. We adopt the convention of summing over repeated indices. Let  $M(t)$  be the Morawetz action

$$M(t) = \int_{\mathbf{R}^d \times \mathbf{R}^d} \frac{(x-y)_j}{|x-y|} \operatorname{Im}[\bar{u}(t, x) \bar{u}(t, y) \partial_j(u(t, x) u(t, y))] dx dy.\tag{4.1}$$

[8] proved

$$\|u\|_{L_{t,x}^4(I \times \mathbf{R}^3)}^4 \lesssim \int_I \partial_t M(t) dt \lesssim \sup_{t \in I} |M(t)| \lesssim \|u\|_{L_t^\infty L_x^2(I \times \mathbf{R}^3)}^3 \|u\|_{L_t^\infty \dot{H}_x^1(I \times \mathbf{R}^3)}.\tag{4.2}$$

[32] proved

$$\begin{aligned} \int_I \int_{\mathbf{R}^d \times \mathbf{R}^d} (-\Delta \Delta |x - y|) |u(t, x)|^2 |u(t, y)|^2 dx dy dt &\lesssim \int_I \partial_t M(t) dt \\ &\lesssim \sup_{t \in I} |M(t)| \lesssim \|u\|_{L_t^\infty L_x^2(I \times \mathbf{R}^d)}^3 \|u\|_{L_t^\infty \dot{H}_x^1(I \times \mathbf{R}^d)}. \end{aligned} \quad (4.3)$$

Additionally, the quantities  $\|u\|_{L_{t,x}^4(I \times \mathbf{R}^3)}$  and

$$\int_I \int_{\mathbf{R}^d \times \mathbf{R}^d} (-\Delta \Delta |x - y|) |u(t, x)|^2 |u(t, y)|^2 dx dy dt$$

are invariant under the transformation  $u \mapsto e^{ix \cdot \xi(t)} u$ . We will show that  $M(t)$  is also Galilean invariant. See [24] for more information.

Indeed, let

$$\tilde{M}(t) = \int_{\mathbf{R}^d \times \mathbf{R}^d} \frac{(x - y)_j}{|x - y|} \operatorname{Im}[\bar{u}(t, x) \bar{u}(t, y) (\partial_j - i\xi_j(t)) u(t, x) u(t, y)] dx dy. \quad (4.4)$$

Then

$$\begin{aligned} &\int_{\mathbf{R}^d \times \mathbf{R}^d} \frac{(x - y)_j}{|x - y|} \operatorname{Im}[\bar{u}(t, x) \bar{u}(t, y) (i\xi_j(t)) u(t, x) u(t, y)] dx dy \\ &= \int_{\mathbf{R}^d \times \mathbf{R}^d} \xi_j(t) \frac{(x - y)_j}{|x - y|} |u(t, x)|^2 |u(t, y)|^2 dx dy. \end{aligned}$$

Because  $|u(t, x)|^2 |u(t, y)|^2$  is even in  $x - y$  and  $\frac{(x - y)_j}{|x - y|}$  is odd in  $x - y$ ,  $M(t) = \tilde{M}(t)$ .

We will not use these estimates directly, instead, we use a frequency localized interaction Morawetz estimate. [10] introduced a frequency localized version of (4.2) for the energy critical nonlinear Schrödinger equation on  $\mathbf{R}^3$  to prove global well-posedness and scattering. In that case  $u(t) \in \dot{H}^1(\mathbf{R}^3)$ , so the Morawetz estimates were localized to high frequencies. Here  $u(t) \in L^2(\mathbf{R}^3)$ , so we localize to low frequencies. In the energy critical case,  $d = 3$ , the  $L_{t,x}^4$  norm scales like

$$\int_I N(t)^{-1} dt,$$

while in the mass critical case the  $L_{t,x}^4$  norm scales like

$$\int_I N(t)^3 dt.$$

This method also has a great deal in common with the almost Morawetz estimates frequently used in conjunction with the I-method. (See [6], [11], and [15] for the two dimensional case, and [14] in the three dimensional case.)

Let  $C$  be a fixed constant and let  $m(\xi)$  be the smooth, radial Fourier multiplier,

$$m(\xi) = \begin{cases} 1, & |\xi| \leq CK; \\ 0, & |\xi| > 2CK. \end{cases} \quad (4.5)$$

**Theorem 4.1** *Suppose  $J$  is a compact interval with  $N(t) \leq 1$  and  $\int_J N(t)^3 dt = K$ . Then if  $u$  is a minimal mass blowup solution to (1.1),  $\mu = +1$ ,*

$$\int_J \int_{\mathbf{R}^d \times \mathbf{R}^d} (-\Delta \Delta |x - y|) |P_{\leq CK} u(t, x)|^2 |P_{\leq CK} u(t, y)|^2 dx dy dt \lesssim_{m_0, d} o(K). \quad (4.6)$$

$o(K)$  is a quantity with  $\lim_{K \rightarrow \infty} \frac{o(K)}{K} = 0$ .

**Remark:** The interaction Morawetz estimates of [32], [24], [?], and [8] rely heavily on  $\mu = +1$ . When  $\mu = -1$  the interaction Morawetz estimates are no longer positive definite, and therefore do not give an estimate of the form (4.2). This is the main obstacle to extending our methods from the defocusing case to the focusing case.

**Remark:** Since  $J$  is a compact interval and  $N(t) \leq 1$ ,

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J \times \mathbf{R}^d)} < \infty.$$

This means  $J$  can be partitioned into a finite number of intervals  $J_k$  with  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} = \epsilon$ .

By lemma 2.7,

$$\sum_{J_k} N(J_k) \sim \int_J N(t)^3 dt.$$

Therefore theorem 4.1 is good enough to exclude the scenario  $\int_0^\infty N(t)^3 dt = \infty$ .

**Remark:** For the rest of this section we will simply write  $\lesssim$  and understand that this refers to  $\lesssim_{m_0, d}$ .

**Theorem 4.2** *If theorem 4.1 is true, then there does not exist a minimal mass blowup solution to (1.1) with  $N(t) \leq 1$ ,  $\mu = +1$ , and*

$$\int_0^\infty N(t)^3 dt = \infty.$$

*Proof of Theorem 4.2:* We want  $\|P_{\leq CK}u(t)\|_{L^2(\mathbf{R}^d)}$  to be very close to  $\|u(t)\|_{L^2(\mathbf{R}^d)}$  for all  $t$ . Therefore, make a Galilean transformation so that  $\xi(0) = 0$ . Consider  $d = 3$  and  $d \geq 4$  separately.

**Case 1,  $d = 3$ :** In this case we need a local well-posedness result.

**Lemma 4.3** *Suppose  $J_1$  is an interval with  $\|P_{\leq CK}u\|_{L_{t,x}^{10/3}(J_1 \times \mathbf{R}^3)} = \frac{\epsilon}{2}$ ,  $C$  is very large, and  $|\xi(t)| \lesssim_d K$ . Then  $\|u\|_{L_{t,x}^{10/3}(J_1 \times \mathbf{R}^3)} \leq \frac{3\epsilon}{4}$ .*

*Proof:* Without loss of generality let  $J_1 = [0, T]$ . By Duhamel's formula and Strichartz estimates,

$$\begin{aligned} \|u\|_{S^0(J_1 \times \mathbf{R}^3)} &\lesssim \|u_0\|_{L^2(\mathbf{R}^3)} + \|P_{\leq CK}u\|_{L_{t,x}^{10/3}(J_1 \times \mathbf{R}^3)}^{7/3} + \\ &\quad \|(1 - P_{\leq CK})u\|_{L_t^\infty L_x^2(J_1 \times \mathbf{R}^3)}^{4/3} \|(1 - P_{\leq CK})u\|_{L_t^2 L_x^6(J_1 \times \mathbf{R}^3)} \\ &\lesssim \|u_0\|_{L^2(\mathbf{R}^3)} + \epsilon^{4/d} + \|(1 - P_{\leq CK})u\|_{L_t^\infty L_x^2(J_1 \times \mathbf{R}^3)}^{4/3} \|u\|_{S^0(J_1 \times \mathbf{R}^3)}. \end{aligned} \quad (4.7)$$

Since  $\|u_0\|_{L^2(\mathbf{R}^3)} \lesssim 1$ ,  $\|(1 - P_{\leq CK})u\|_{L_t^\infty L_x^2(J_1 \times \mathbf{R}^3)}$  sufficiently small implies  $\|u\|_{S^0(J \times \mathbf{R}^3)} \lesssim 1$  by continuity. Interpolating  $\|(1 - P_{\leq CK})u\|_{L_t^2 L_x^6(J_1 \times \mathbf{R}^3)} \lesssim 1$  with  $\|(1 - P_{\leq CK})u\|_{L_t^\infty L_x^2(J_1 \times \mathbf{R}^3)} \leq \delta(\epsilon)$  for  $\delta(\epsilon) > 0$  sufficiently small implies  $\|u\|_{L_{t,x}^{10/3}(J_1 \times \mathbf{R}^3)} \leq \frac{3\epsilon}{4}$ . By (2.13),  $|\xi(t)| \lesssim_d K$ , so we can choose  $C(\delta, d)$  sufficiently large so that

$$\|u_{> \frac{CK}{2}}\|_{L_t^\infty L_x^2(J_1 \times \mathbf{R}^3)} \leq \delta(\epsilon).$$

□

**Remark:** By lemma 2.8, if  $\int_J N(t)^3 dt = K$ , then for any  $t_1, t_2 \in J$ ,  $|\xi(t_1) - \xi(t_2)| \lesssim_d K$ . Therefore if  $\int_J N(t)^3 dt = K$  we can make a Galilean transformation so that  $|\xi(t)| \lesssim_d K$  on  $J$ .

Now take a subinterval  $J_k$  with  $\|u\|_{L_{t,x}^{10/3}(J_k \times \mathbf{R}^3)} = \epsilon$ . Lemma 4.3 implies that  $\|P_{\leq CK}u\|_{L_{t,x}^{10/3}(J_k \times \mathbf{R}^3)} \geq \frac{\epsilon}{2}$ . From (2.20),

$$\int_{J_k} N(t)^2 dt \lesssim \int_{J_k} \int_{\mathbf{R}^3} |u(t, x)|^{10/3} dx dt \lesssim \epsilon^{10/3}. \quad (4.8)$$

By lemma 2.5,  $N(t_1) \sim N(t_2)$  on  $J_k$ , so

$$|J_k| \lesssim \frac{\epsilon^{10/3}}{N(J_k)^2}.$$

By Holder's inequality,

$$\|P_{\leq CK}u\|_{L_t^{8/3} L_x^4(J_k \times \mathbf{R}^3)} \lesssim \left(\frac{1}{N(J_k)^2}\right)^{1/8} \|P_{\leq CK}u\|_{L_{t,x}^4(J_k \times \mathbf{R}^d)}. \quad (4.9)$$



This implies

$$N(J_k) \|P_{\leq CK} u\|_{L_t^{8/3} L_x^4(J_k \times \mathbf{R}^3)}^4 \lesssim \|P_{\leq CK} u\|_{L_{t,x}^4(J_k \times \mathbf{R}^3)}^4. \quad (4.10)$$

By interpolation if  $\|P_{\leq CK} u\|_{L_{t,x}^{10/3}(J \times \mathbf{R}^3)} \geq \frac{\epsilon}{2}$  and  $\|P_{\leq CK} u\|_{L_t^\infty L_x^2(J_k \times \mathbf{R}^3)} \lesssim 1$ , then  $\|P_{\leq CK} u\|_{L_t^{8/3} L_x^4(J_k \times \mathbf{R}^3)} \gtrsim \epsilon^{5/4}$ , so

$$\begin{aligned} \int_J N(t)^3 dt &\sim \sum_{J_k} N(J_k) \lesssim \sum_{J_k} N(J_k) \|P_{\leq CK} u\|_{L_{t,x}^{8/3}(J_k \times \mathbf{R}^3)}^4 \\ &\lesssim \sum_{J_k} \|P_{\leq CK} u\|_{L_{t,x}^4(J_k \times \mathbf{R}^3)}^4 = \int_J \int_{\mathbf{R}^3} |P_{\leq CK} u(t, x)|^4 dx dt. \end{aligned} \quad (4.11)$$

When  $d = 3$ ,

$$(-\Delta \Delta |x - y|) = 4\pi \delta(|x - y|).$$

Therefore

$$\int_{\mathbf{R}^3 \times \mathbf{R}^3} (-\Delta \Delta |x - y|) |P_{\leq CK} u(t, y)|^2 |P_{\leq CK} u(t, x)|^2 dx dy = \int_{\mathbf{R}^3} |P_{\leq CK} u(t, x)|^4 dx.$$

Now if

$$\int_0^T N(t)^3 dt = K,$$

then by theorem 4.1,

$$K \lesssim_d \int_0^T \int_{\mathbf{R}^3} |Iu(t, x)|^4 dx dt \lesssim_d o(K). \quad (4.12)$$

This gives a contradiction if  $K$  is sufficiently large. When  $\int_0^\infty N(t)^3 dt = \infty$  we can always a suitable  $T$ .

**Case 2,  $d \geq 4$ :**

$$\begin{aligned} &\int_J \int_{\mathbf{R}^d \times \mathbf{R}^d} (-\Delta \Delta |x - y|) |P_{\leq CK} u(t, x)|^2 |P_{\leq CK} u(t, y)|^2 dx dy dt \\ &= \int_J \int_{\mathbf{R}^d \times \mathbf{R}^d} \left( \frac{4(d-1)(d-3)}{|x - y|^3} \right) |P_{\leq CK} u(t, x)|^2 |P_{\leq CK} u(t, y)|^2 dx dy dt. \end{aligned} \quad (4.13)$$

Let  $\eta = \frac{m_0^2}{1000}$ .

$$\int_{|x-x(t)| \leq \frac{C(\eta)}{N(t)}} |u(t, x)|^2 dx \geq m_0^2 - \eta. \quad (4.14)$$

Also,

$$\int_{|\xi - \xi(t)| > \frac{C(\eta)N(t)}{2}} |\hat{u}(t, \xi)|^2 d\xi \leq \eta. \quad (4.15)$$

Therefore, for  $K \geq 1$ ,  $C$  sufficiently large,

$$\int_{|x - x(t)| \leq \frac{C(\eta)}{N(t)}} |P_{\leq CK} u(t, x)|^2 dx \geq \frac{m_0^2}{2}. \quad (4.16)$$

Of course, for the same  $x(t) \in \mathbf{R}^d$  we also have

$$\int_{|y - x(t)| \leq \frac{C(\eta)}{N(t)}} |P_{\leq CK} u(t, y)|^2 dy \geq \frac{m_0^2}{2}. \quad (4.17)$$

Therefore, because  $N(t) \leq 1$ ,

$$\begin{aligned} N(t)^3 &\lesssim N(t)^3 \left( \int_{|x - x(t)| \leq \frac{C(\eta)}{N(t)}} |P_{\leq CK} u(t, x)|^2 dx \right) \left( \int_{|y - x(t)| \leq \frac{C(\eta)}{N(t)}} |P_{\leq CK} u(t, y)|^2 dy \right) \\ &\lesssim N(t)^3 \int_{|x - y| \leq \frac{2C(\eta)}{N(t)}} |P_{\leq CK} u(t, x)|^2 |P_{\leq CK} u(t, y)|^2 dx dy \\ &\lesssim \int_{\mathbf{R}^d \times \mathbf{R}^d} \frac{1}{|x - y|^3} |P_{\leq CK} u(t, x)|^2 |P_{\leq CK} u(t, y)|^2 dx dy. \end{aligned}$$

Once again, this implies that for a compact interval  $J$ ,

$$K = \int_J N(t)^3 dt \lesssim_d \int_J \int_{\mathbf{R}^d \times \mathbf{R}^d} \left( \frac{1}{|x - y|} \right)^3 |P_{\leq CK} u(t, x)|^2 |P_{\leq CK} u(t, y)|^2 dx dy dt \lesssim o(K). \quad (4.18)$$

This gives a contradiction for  $K$  sufficiently large.  $\square$

All that is left to do is to prove theorem 4.1, which will occupy the remainder of the section. We begin by estimating the error for the truncated Morawetz estimates. For the rest of the section  $C(\epsilon, m_0, d)$  will be a fixed constant so that (4.16) is satisfied,  $|\xi(t)| \leq \frac{CK}{1000}$  on  $J$  if  $\int_J N(t)^3 dt = K$ , and  $\|P_{\leq CK} u\|_{L_{t,x}^{10/3}(J_1 \times \mathbf{R}^3)} \leq \frac{\epsilon}{2}$  implies  $\|u\|_{L_{t,x}^{10/3}(J_1 \times \mathbf{R}^3)} \leq \frac{3\epsilon}{4}$ .

**Theorem 4.4** *Let  $a(x, y) = |x - y|$ . Define the interaction Morawetz quantity*

$$M(t) = \int a_j(x, y) \operatorname{Im} [\overline{P_{\leq CK} u(t, x) P_{\leq CK} u(t, y)} (\partial_j - i\xi_j(t)) (P_{\leq CK} u(t, x) P_{\leq CK} u(t, y))] dx dy. \quad (4.19)$$

Then for  $\mu = +1$ ,

$$\int_0^T \int_{\mathbf{R}^d \times \mathbf{R}^d} (-\Delta \Delta |x - y|) |P_{\leq CK} u(t, x)|^2 |P_{\leq CK} u(t, y)|^2 dx dy dt \lesssim o(K). \quad (4.20)$$

**Remark:** We adopt the usual convention of summing over repeated indices.

*Proof:* First take  $M(t)$ .

$$\int_{\mathbf{R}^d \times \mathbf{R}^d} \frac{(x - y)_j}{|x - y|} \operatorname{Im} [\overline{P_{\leq CK} u}(t, x) \overline{P_{\leq CK} u}(t, y) (\partial_j - i\xi_j(t)) P_{\leq CK} u(t, x) P_{\leq CK} u(t, y)] dx dy$$

$$\lesssim \|P_{\leq CK} u\|_{L_t^\infty L_x^2([0, T] \times \mathbf{R}^d)}^3 \|(\nabla - i\xi(t)) P_{\leq CK} u\|_{L_t^\infty L_x^2([0, T] \times \mathbf{R}^d)} \lesssim o(K).$$

We estimate  $\|P_{\leq CK} u\|_{L_t^\infty L_x^2([0, T] \times \mathbf{R}^d)}$  by conservation of mass and  $\|(\nabla - i\xi(t)) P_{\leq CK} u\|_{L_t^\infty L_x^2([0, T] \times \mathbf{R}^d)}$  by (2.13) and  $N(t) \leq 1$ .

Since  $P_{\leq CK}$  is a Fourier multiplier,

$$\partial_t(P_{\leq CK} u) = i\Delta P_{\leq CK} u - i|P_{\leq CK} u|^{4/d}(P_{\leq CK} u) + i|P_{\leq CK} u|^{4/d}(P_{\leq CK} u) - iP_{\leq CK}(|u|^{4/d}u). \quad (4.21)$$

If we had only

$$\partial_t(P_{\leq CK} u) = i\Delta(P_{\leq CK} u) - i|P_{\leq CK} u|^{4/d}(P_{\leq CK} u)$$

then the proof of theorem 4.4 would be complete. We could copy the arguments from [8] and [32] exactly, replacing  $u$  with  $P_{\leq CK} u$ . Instead, it is necessary to deal with the error terms that arise from the fact that  $|P_{\leq CK} u|^{4/d}(P_{\leq CK} u) - P_{\leq CK}(|u|^{4/d}u) \neq 0$ , and prove these error terms are  $\lesssim o(K)$ . Let  $x$  denote the first  $d$  variables in  $\mathbf{R}^d \times \mathbf{R}^d$  and  $y$  the second  $d$  variables. We have the error

$$\begin{aligned} \mathcal{E} &= \int_0^T \int_{\mathbf{R}^d \times \mathbf{R}^d} a_j(x, y) |P_{\leq CK} u(t, y)|^2 \\ &\operatorname{Re}\{[P_{\leq CK}(|u|^{4/d}\bar{u})(t, x) - |P_{\leq CK} u|^{4/d}(\overline{P_{\leq CK} u})(t, x)](\partial_j - i\xi_j(t)) P_{\leq CK} u(t, x)\} dx dy dt \end{aligned} \quad (4.22)$$

$$\begin{aligned} &+ \int_0^T \int_{\mathbf{R}^d \times \mathbf{R}^d} a_j(x, y) |P_{\leq CK} u(t, y)|^2 \\ &\operatorname{Re}\{\overline{P_{\leq CK} u}(t, x) (\partial_j - i\xi_j(t)) [|P_{\leq CK} u|^{4/d}(P_{\leq CK} u)(t, x) - P_{\leq CK}(|u|^{4/d}u)(t, x)]\} dx dy dt \end{aligned} \quad (4.23)$$

$$\begin{aligned}
& + \int_0^T \int_{\mathbf{R}^d \times \mathbf{R}^d} a_j(x, y) \operatorname{Re}[\overline{P_{\leq CK} u}(t, x) (\partial_j - i\xi_j(t)) P_{\leq CK} u(t, x)] \\
& \quad [P_{\leq CK}(|u|^{4/d} \bar{u})(t, y) P_{\leq CK} u(t, y) - P_{\leq CK}(|u|^{4/d} u)(t, y) \overline{P_{\leq CK} u}(t, y)] dx dy dt.
\end{aligned} \tag{4.24}$$

Now we need some intermediate lemmas.

**Lemma 4.5** *Suppose  $u$  satisfies*

$$\|P_{|\xi - \xi(t)| > N} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0, T] \times \mathbf{R}^d)} \lesssim_{m_0, d} \rho(N) \left(\left(\frac{K}{N}\right)^{1/2} + 1\right), \tag{4.25}$$

$\rho(N) \leq 1$ ,  $\rho(N) \rightarrow 0$  as  $N \rightarrow \infty$ ,  $|\xi(t)| \leq \frac{CK}{1000}$ . Then for any  $1/2 < s \leq 1$ ,

$$\| |\nabla|^s e^{-ix \cdot \xi(t)} P_{\leq CK} u \|_{L_t^2 L_x^{\frac{2d}{d-2}}([0, T] \times \mathbf{R}^d)} \lesssim o(K^s). \tag{4.26}$$

*Proof:*

$$\begin{aligned}
& \| |\nabla|^s (e^{-ix \cdot \xi(t)} P_{\leq CK} u) \|_{L_t^2 L_x^{\frac{2d}{d-2}}([0, T] \times \mathbf{R}^d)} \lesssim \sum_{N \leq 2CK} N^s \|P_N(e^{-ix \cdot \xi(t)} P_{\leq CK} u)\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0, T] \times \mathbf{R}^d)} \\
& \lesssim \sum_{N \leq 2CK} N^s \|P_N(e^{-ix \cdot \xi(t)} u)\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0, T] \times \mathbf{R}^d)} \lesssim \sum_{N \leq 2CK} N^s \rho(N) \left(\frac{K}{N}\right)^{1/2} \lesssim o(K^s).
\end{aligned}$$

□

**Lemma 4.6** *Suppose  $u$  satisfies the hypotheses of lemma 4.5. Then for  $1/2 < s \leq 1$ ,*

$$\| |\nabla|^s (e^{-ix \cdot \xi(t)} P_{\leq CK}(|u|^{4/d} u)) \|_{L_t^2 L_x^{\frac{2d}{d+2}}([0, T] \times \mathbf{R}^d)} \lesssim_{m_0, d} K^s. \tag{4.27}$$

*Proof:* Again make a Littlewood-Paley decomposition.

$$\begin{aligned}
& \| |\nabla|^s (e^{-ix \cdot \xi(t)} P_{\leq CK}(|u|^{4/d} u)) \|_{L_t^2 L_x^{\frac{2d}{d+2}}([0, T] \times \mathbf{R}^d)} \\
& \leq \sum_{N \leq 2CK} N^s \|P_N(e^{-ix \cdot \xi(t)} P_{\leq CK}(|u|^{4/d} u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0, T] \times \mathbf{R}^d)} \\
& \quad \sum_{N \leq \frac{CK}{4}} N^s \|P_N(e^{-ix \cdot \xi(t)} P_{\leq CK}(|u|^{4/d} u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0, T] \times \mathbf{R}^d)} \\
& = \sum_{N \leq \frac{CK}{4}} N^s \|P_N(e^{-ix \cdot \xi(t)} (|u|^{4/d} u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0, T] \times \mathbf{R}^d)}.
\end{aligned} \tag{4.28}$$

By Bernstein's inequality,

$$\begin{aligned} & \|P_N(|P_{\leq N}(e^{-ix \cdot \xi(t)}u)|^{4/d}(P_{\leq N}(e^{-ix \cdot \xi(t)}u)))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \\ & \lesssim \frac{1}{N} \|u\|_{L_t^\infty L_x^2([0,T] \times \mathbf{R}^d)}^{4/d} \|\nabla(P_{\leq N}(e^{-ix \cdot \xi(t)}u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \lesssim \frac{K^{1/2}}{N^{1/2}}. \end{aligned}$$

By Holder's inequality, conservation of mass,

$$\|P_{>N}(e^{-ix \cdot \xi(t)}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \|u\|_{L_t^\infty L_x^2([0,T] \times \mathbf{R}^d)}^{4/d} \lesssim \frac{K^{1/2}}{N^{1/2}}.$$

Therefore, for  $N \leq \frac{CK}{4}$ ,

$$\|P_N(e^{-ix \cdot \xi(t)}P_{\leq CK}(|u|^{4/d}u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \lesssim \frac{K^{1/2}}{N^{1/2}}.$$

Meanwhile,

$$\begin{aligned} & \|P_{\geq \frac{CK}{4}}(e^{-ix \cdot \xi(t)}P_{\leq CK}(|u|^{4/d}u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \\ & \leq \|P_{\geq \frac{CK}{5}}(|u|^{4/d}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \\ & \leq \|P_{\geq \frac{CK}{8}}(e^{-ix \cdot \xi(t)}|u|^{4/d}(e^{-ix \cdot \xi(t)}u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)}. \end{aligned}$$

Again combining Bernstein's inequality, conservation of mass, and Holder's inequality,

$$\|P_{\geq \frac{CK}{8}}(e^{-ix \cdot \xi(t)}|u|^{4/d}(e^{-ix \cdot \xi(t)}u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \lesssim 1.$$

Therefore (4.28)  $\lesssim K^s$ .  $\square$

**Lemma 4.7** *Suppose  $u$  satisfies the hypotheses of lemma 4.5. Then*

$$\|P_{\leq CK}(|u|^{4/d}u) - |P_{\leq CK}u|^{4/d}(P_{\leq CK}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \lesssim_{m_0,d} 1. \quad (4.29)$$

*Proof:* By lemma 4.6,

$$\|\nabla e^{-ix \cdot \xi(t)}P_{\leq CK}(|u|^{4/d}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \lesssim K.$$

Also, by the chain rule and conservation of mass,

$$\|\nabla e^{-ix \cdot \xi(t)} (|P_{\leq CK} u|^{4/d} (P_{\leq CK} u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \lesssim_{m_0,d} \|\nabla e^{-ix \cdot \xi(t)} P_{\leq CK} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T] \times \mathbf{R}^d)}.$$

Since  $P_{\leq CK} = 1$  on  $|\xi| \leq CK$ ,

$$\|\nabla e^{-ix \cdot \xi(t)} P_{\leq CK} (P_{\leq \frac{CK}{4}} u)\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T] \times \mathbf{R}^d)} = \|\nabla P_{\leq CK} e^{-ix \cdot \xi(t)} (P_{\leq \frac{CK}{4}} u)\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T] \times \mathbf{R}^d)} \lesssim K.$$

The last inequality follows from lemma 4.5.

$$\|\nabla e^{-ix \cdot \xi(t)} P_{\leq CK} (P_{\geq \frac{CK}{4}} u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \lesssim K \|u_{|\xi - \xi(t)| > \frac{CK}{8}}\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \lesssim K.$$

Therefore, by Bernstein's inequality,

$$\|P_{> \frac{CK}{4}} e^{-ix \cdot \xi(t)} [P_{\leq CK} (|u|^{4/d} u) - |P_{\leq CK} u|^{4/d} (P_{\leq CK} u)]\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \lesssim 1. \quad (4.30)$$

On the other hand, by  $|\xi(t)| \leq \frac{CK}{1000}$  and Holder's inequality,

$$\begin{aligned} & \|P_{\leq \frac{CK}{4}} e^{-ix \cdot \xi(t)} [P_{\leq CK} (|u|^{4/d} u) - |P_{\leq CK} u|^{4/d} (P_{\leq CK} u)]\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \\ & \leq \| |u|^{4/d} u - |P_{\leq CK} u|^{4/d} (P_{\leq CK} u) \|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \\ & \lesssim \| |P_{> \frac{CK}{4}} u| |u|^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \lesssim \|u_{|\xi - \xi(t)| \geq \frac{CK}{8}}\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T] \times \mathbf{R}^d)} \lesssim 1. \end{aligned}$$

Therefore the proof is complete.  $\square$

We are now ready to estimate the first term in  $\mathcal{E}$ .

**Corollary 4.8**

$$(4.22) \lesssim o(K). \quad (4.31)$$

*Proof:* Because  $\frac{(x-y)_j}{|x-y|}$  is uniformly bounded on  $\mathbf{R}^d \times \mathbf{R}^d$ , by lemmas 4.5, 4.7,

$$\begin{aligned} (4.22) & \lesssim \|P_{\leq CK} u\|_{L_t^\infty L_x^2([0,T] \times \mathbf{R}^d)}^2 \|e^{ix \cdot \xi(t)} \nabla (e^{-ix \cdot \xi(t)} P_{\leq CK} u)\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T] \times \mathbf{R}^d)} \\ & \quad \times \|P_{\leq CK} (|u|^{4/d} u) - |P_{\leq CK} u|^{4/d} (P_{\leq CK} u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \lesssim o(K). \end{aligned}$$

$\square$

In order to estimate (4.47) and (4.48) we need one additional lemma.

**Lemma 4.9** Suppose  $K(x)$  is a kernel,

$$|K(x)| \lesssim_d 1, \quad (4.32)$$

and

$$|\nabla K(x)| \lesssim_d \frac{1}{|x|}. \quad (4.33)$$

Let

$$F(x) = \int K(x-y) \cdot (\nabla f(y))g(y)dy. \quad (4.34)$$

Then  $F(x) = G(x) + H(x)$ , where for  $\frac{1}{p} + \frac{1}{p'} = 1$ ,

$$\|G\|_{L_x^\infty(\mathbf{R}^d)} \lesssim_d \|\nabla g\|_{L_x^p(\mathbf{R}^d)} \|f\|_{L_x^{p'}(\mathbf{R}^d)}, \quad (4.35)$$

$$\|H\|_{L_x^{\frac{6d}{5}}(\mathbf{R}^d)} \lesssim_d \| |\nabla|^{2/3} g \|_{L_x^{\frac{2d}{d-1}}(\mathbf{R}^d)} \|f\|_{L_x^{\frac{2d}{d+2}}(\mathbf{R}^d)}, \quad (4.36)$$

and

$$\|H\|_{L_x^{3d}(\mathbf{R}^d)} \lesssim_d \| |\nabla|^{2/3} g \|_{L_x^{\frac{2d}{d+2}}(\mathbf{R}^d)} \|f\|_{L_x^{\frac{2d}{d-2}}(\mathbf{R}^d)}. \quad (4.37)$$

*Proof:* This is proved by integration by parts and the Hardy-Littlewood-Sobolev inequality.

$$\int K(x-y) \cdot (\nabla f(y))g(y)dy = - \int K(x-y) \cdot (\nabla g(y))f(y)dy - \int (\nabla \cdot K(x-y))g(y)f(y)dy.$$

Let

$$G(x) = - \int K(x-y) \cdot (\nabla g(y))f(y)dy$$

and

$$H(x) = - \int (\nabla \cdot K(x-y))g(y)f(y)dy.$$

Apply Holder's inequality and  $|K(x-y)| \lesssim_d 1$  to  $G(x)$  and the Hardy-Littlewood-Sobolev inequality,  $|\nabla K(x-y)| \lesssim_d \frac{1}{|x-y|}$ , and the Sobolev embedding theorem to  $H(x)$ .  $\square$

**Corollary 4.10**

$$(4.47) \lesssim o(K).$$

*Proof:* Let

$$F(t, y) = \int_{\mathbf{R}^d} \frac{(x-y)_j}{|x-y|} \partial_j (e^{-ix \cdot \xi(t)} [P_{\leq CK}(|u|^{4/d}u)(t, x) - |P_{\leq CK}u|^{4/d}(P_{\leq CK}u)(t, x)]) (e^{ix \cdot \xi(t)} \overline{P_{\leq CK}u}(t, x)) dx.$$

Then by lemma 4.9,

$$F(t, y) = G(t, y) + H(t, y), \quad (4.38)$$

with

$$\begin{aligned} \|G(t, y)\|_{L_t^1 L_x^\infty([0, T] \times \mathbf{R}^d)} &\lesssim \|P_{\leq CK}(|u|^{4/d}u) - |P_{\leq CK}u|^{4/d}(P_{\leq CK}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0, T] \times \mathbf{R}^d)} \\ &\quad \times \|\nabla e^{ix \cdot \xi(t)} P_{\leq CK}u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0, T] \times \mathbf{R}^d)} \lesssim o(K), \end{aligned} \quad (4.39)$$

and

$$\begin{aligned} \|H(t, y)\|_{L_t^{4/3} L_x^{\frac{6d}{5}}([0, T] \times \mathbf{R}^d)} &\lesssim \|P_{\leq CK}(|u|^{4/d}u) - |P_{\leq CK}u|^{4/d}(P_{\leq CK}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0, T] \times \mathbf{R}^d)} \\ &\quad \|\nabla|^{2/3} e^{ix \cdot \xi(t)} P_{\leq CK}u\|_{L_t^4 L_x^{\frac{2d}{d-1}}([0, T] \times \mathbf{R}^d)} \lesssim o(K^{2/3}). \end{aligned} \quad (4.40)$$

By Holder's inequality, conservation of mass,

$$\int_0^T \int_{\mathbf{R}^d} |P_{\leq CK}u(t, y)|^2 |G(t, y)| dy dt \leq \|G(t, y)\|_{L_t^1 L_x^\infty([0, T] \times \mathbf{R}^d)} \|P_{\leq CK}u(t, y)\|_{L_t^\infty L_x^2([0, T] \times \mathbf{R}^d)}^2 \lesssim o(K).$$

By Sobolev embedding, lemma 4.5,

$$\begin{aligned} &\int_0^T \int_{\mathbf{R}^d} |P_{\leq CK}u(t, y)|^2 |H(t, y)| dy dt \\ &\leq \|H(t, y)\|_{L_t^{4/3} L_x^{\frac{6d}{5}}([0, T] \times \mathbf{R}^d)} \|P_{\leq CK}u(t, y)\|_{L_t^8 L_x^{\frac{12d}{6d-5}}([0, T] \times \mathbf{R}^d)}^2 \lesssim o(K). \end{aligned}$$

This implies (4.47)  $\lesssim o(K)$ .  $\square$

Finally consider (4.48).

$$\begin{aligned} P_{\leq CK}(|u|^{4/d}u) \overline{P_{\leq CK}u} &= |u|^{2+4/d} + (P_{\leq CK} - 1)(|u|^{4/d}u) (\overline{P_{\leq CK}u}) \\ &+ (1 - P_{\leq CK})(|u|^{4/d}u) \overline{(P_{\leq CK} - 1)u} + P_{\leq CK}(|u|^{4/d}u) \overline{(P_{\leq CK} - 1)u}. \end{aligned}$$



$$Im[|u|^{2+4/d}] \equiv 0.$$

Next, let

$$F_{1j}(t, x) = \int_{\mathbf{R}^d} \frac{(x-y)_j}{|x-y|} (1 - P_{\leq CK}) (|u|^{4/d} u)(t, y) \overline{(P_{\leq CK} - 1)u}(t, y) dy, \quad (4.41)$$

$$F_{2j}(t, x) = \int_{\mathbf{R}^d} \frac{(x-y)_j}{|x-y|} (P_{\leq CK} - 1) (|u|^{4/d} u)(t, y) \overline{(P_{\leq CK} - 1)u}(t, y) dy, \quad (4.42)$$

and

$$F_{3j}(t, x) = \int_{\mathbf{R}^d} \frac{(x-y)_j}{|x-y|} P_{\leq CK} (|u|^{4/d} u)(t, y) \overline{(P_{\leq CK} - 1)u}(t, y) dy. \quad (4.43)$$

By Holder's inequality, lemma 4.5, lemma 4.6, and  $|\xi - \xi(t)| \sim |\xi|$  for  $|\xi| \geq CK$ ,

$$\begin{aligned} & \|F_{1j}\|_{L_t^1 L_x^\infty([0, T] \times \mathbf{R}^d)} \\ & \lesssim \|(1 - P_{\leq CK})u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0, T] \times \mathbf{R}^d)} \|(1 - P_{\leq CK})(|u|^{4/d} u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0, T] \times \mathbf{R}^d)} \lesssim o(1). \end{aligned}$$

Next, by lemma 4.9,  $e^{-ix \cdot \xi(t)} (P_{\leq CK} - 1)(|u|^{4/d} u) = \nabla \cdot \frac{\nabla}{\Delta} e^{-ix \cdot \xi(t)} (P_{\leq CK} - 1)(|u|^{4/d} u)$ , we have  $F_{2j} = G_{2j} + H_{2j}$ , where

$$\begin{aligned} & \|G_{2j}\|_{L_t^1 L_x^\infty([0, T] \times \mathbf{R}^d)} \\ & \lesssim \|\nabla e^{-ix \cdot \xi(t)} P_{\leq CK} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0, T] \times \mathbf{R}^d)} \left\| \frac{\nabla}{\Delta} e^{-ix \cdot \xi(t)} (P_{\leq CK} - 1)(|u|^{4/d} u) \right\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0, T] \times \mathbf{R}^d)} \\ & \lesssim o(K) \left( \frac{1}{K} \right) = o(1). \end{aligned}$$

We use the fact that  $|\xi - \xi(t)| \gtrsim K$  on the support of  $(1 - P_{\leq CK})$ .

$$\begin{aligned} & \|H_{2j}\|_{L_t^{4/3} L_x^{\frac{6d}{5}}([0, T] \times \mathbf{R}^d)} \\ & \lesssim \| |\nabla|^{2/3} e^{-ix \cdot \xi(t)} P_{\leq CK} u \|_{L_t^2 L_x^{\frac{2d}{d-2}}([0, T] \times \mathbf{R}^d)} \left\| \frac{\nabla}{\Delta} e^{-ix \cdot \xi(t)} (P_{\leq CK} - 1)(|u|^{4/d} u) \right\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0, T] \times \mathbf{R}^d)} \lesssim o(K^{-1/3}). \end{aligned}$$

Finally,  $F_{3j} = G_{3j} + H_{3j}$ , with

$$\begin{aligned} & \|G_{3j}\|_{L_t^1 L_x^\infty([0, T] \times \mathbf{R}^d)} \\ & \lesssim \|\nabla e^{-ix \cdot \xi(t)} P_{\leq CK} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0, T] \times \mathbf{R}^d)} \left\| \frac{\nabla}{\Delta} e^{-ix \cdot \xi(t)} (P_{\leq CK} - 1)(|u|^{4/d} u) \right\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0, T] \times \mathbf{R}^d)} \lesssim o(1), \end{aligned}$$

$$\begin{aligned}
& \|H_{3j}\|_{L_t^1 L_x^{3d}([0,T] \times \mathbf{R}^d)} \\
& \lesssim \| |\nabla|^{2/3} e^{-ix \cdot \xi(t)} P_{\leq CK} u \|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T] \times \mathbf{R}^d)} \left\| \frac{\nabla}{\Delta} e^{-ix \cdot \xi(t)} (P_{\leq CK} - 1)(|u|^{4/d} u) \right\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T] \times \mathbf{R}^d)} \lesssim o(K^{-1/3}).
\end{aligned}$$

By Holder's inequality,

$$\begin{aligned}
& \int_0^T \int_{\mathbf{R}^d} |F_{1j}(t, x) + G_{2j}(t, x) + G_{3j}(t, x)| |(\nabla - i\xi(t)) P_{\leq CK} u(t, x)| |P_{\leq CK} u(t, x)| dx dt \\
& \lesssim \|F_{1j}(t, x) + G_{2j}(t, x) + G_{3j}(t, x)\|_{L_t^1 L_x^\infty([0,T] \times \mathbf{R}^d)} \\
& \quad \times \|\nabla e^{-ix \cdot \xi(t)} P_{\leq CK} u\|_{L_t^\infty L_x^2([0,T] \times \mathbf{R}^d)} \|P_{\leq CK} u(t, x)\|_{L_t^\infty L_x^2([0,T] \times \mathbf{R}^d)} \lesssim o(K).
\end{aligned}$$

Next, by Holder's inequality and Sobolev embedding,

$$\begin{aligned}
& \int_0^T \int_{\mathbf{R}^d} |H_{3j}(t, x)| |\nabla e^{-ix \cdot \xi(t)} P_{\leq CK} u(t, x)| |P_{\leq CK} u(t, x)| dx dt \\
& \lesssim \|H_{3j}\|_{L_t^1 L_x^{3d}([0,T] \times \mathbf{R}^d)} \|\nabla e^{-ix \cdot \xi(t)} P_{\leq CK} u(t, x)\|_{L_t^\infty L_x^2([0,T] \times \mathbf{R}^d)} \|e^{-ix \cdot \xi(t)} P_{\leq CK} u\|_{L_t^\infty L_x^{\frac{6d}{3d-2}}([0,T] \times \mathbf{R}^d)} \\
& \lesssim o(K^{-1/3}) K K^{1/3} = o(K).
\end{aligned}$$

Finally, by the Sobolev embedding theorem, lemma 4.5, and interpolation,

$$\begin{aligned}
& \int_0^T \int_{\mathbf{R}^d} |H_{2j}(t, x)| |\nabla e^{-ix \cdot \xi(t)} P_{\leq CK} u(t, x)| |P_{\leq CK} u(t, x)| dx dt \\
& \lesssim \|H_{2j}\|_{L_t^{4/3} L_x^{\frac{6d}{5}}([0,T] \times \mathbf{R}^d)} \|\nabla e^{-ix \cdot \xi(t)} P_{\leq CK} u\|_{L_t^4 L_x^{\frac{2d}{d-1}}([0,T] \times \mathbf{R}^d)} \|P_{\leq CK} u\|_{L_t^\infty L_x^{\frac{6d}{3d-2}}([0,T] \times \mathbf{R}^d)} \\
& \lesssim o(K^{-1/3}) K K^{1/3} = o(K).
\end{aligned}$$

This completes the proof of theorem 4.4.  $\square$

Therefore, scenario  $\int_0^\infty N(t)^3 dt = \infty$  has been excluded.

We have actually proved a more general estimate.

**Theorem 4.11** Suppose  $a_j(t, x)$  is an odd function on  $\mathbf{R}^d$  for all  $t$  and there exists a constant  $C$  such that

$$|a_j(t, x)| \leq C, \quad (4.44)$$

$$|\partial_k a_j(t, x)| \leq \frac{C}{|x|}. \quad (4.45)$$

Suppose also that  $u(t, x)$  is a minimal mass blowup solution to (1.1),  $\mu = \pm 1$ . Then

$$\begin{aligned} \int_0^T \int_{\mathbf{R}^d \times \mathbf{R}^d} a_j(t, x-y) |P_{\leq CK} u(t, y)|^2 \operatorname{Re}\{[P_{\leq CK}(|u|^{4/d} \bar{u})(t, x) - |P_{\leq CK} u|^{4/d}(\overline{P_{\leq CK} u})(t, x)] \\ \times (\partial_j - i\xi_j(t)) P_{\leq CK} u(t, x)\} dx dy dt \lesssim_{m_0, d} o(K)C, \end{aligned} \quad (4.46)$$

$$\begin{aligned} \int_0^T \int_{\mathbf{R}^d \times \mathbf{R}^d} a_j(t, x-y) |P_{\leq CK} u(t, y)|^2 \operatorname{Re}\{\overline{P_{\leq CK} u}(t, x) (\partial_j - i\xi_j(t)) \\ \times [|P_{\leq CK} u|^{4/d}(\overline{P_{\leq CK} u})(t, x) - P_{\leq CK}(|u|^{4/d} u)(t, x)]\} dx dy dt \lesssim_{m_0, d} o(K)C, \end{aligned} \quad (4.47)$$

$$\begin{aligned} \int_0^T \int_{\mathbf{R}^d \times \mathbf{R}^d} a_j(t, x-y) \operatorname{Re}[\overline{P_{\leq CK} u}(t, x) (\partial_j - i\xi_j(t)) P_{\leq CK} u(t, x)] \\ [P_{\leq CK}(|u|^{4/d} \bar{u})(t, y) P_{\leq CK} u(t, y) - P_{\leq CK}(|u|^{4/d} u)(t, y) \overline{P_{\leq CK} u}(t, y)] dx dy dt \lesssim_{m_0, d} o(K)C. \end{aligned} \quad (4.48)$$

*Proof:* By theorem 3.1 a minimal mass blowup solution to (1.1),  $\mu = \pm 1$  satisfies the hypotheses of lemma 4.5.  $\square$

**Remark:** We conclude this section with a brief summary of what we have done. We have excluded the scenario when  $\mu = +1$ ,  $\int_0^\infty N(t)^3 dt = \infty$  by proving that the errors arising from the interaction Morawetz estimates (4.2), (4.3) are bounded by  $o(K)$ . In the defocusing case these interaction Morawetz estimates are positive definite and  $\gtrsim K$ , which is a contradiction for  $K$  sufficiently large. In the focusing case (4.2) and (4.3) are not positive definite. However, theorem 4.11 states that if we did find an appropriate interaction Morawetz potential that satisfies (4.44), (4.45), then the error would be bounded by  $o(K)$ .

## 5 $\int_0^\infty N(t)^3 dt < \infty$

In this section we exclude the existence of a minimal mass blowup solution with  $N(t) \leq 1$  and

$$\int_0^\infty N(t)^3 dt = K < \infty. \quad (5.1)$$

Excluding this scenario concludes the proof of theorem 1.6. As in [31] and [23] we will prove additional regularity. Conservation of energy precludes  $N(t) \rightarrow 0$  as  $t \rightarrow \infty$ , giving a contradiction. To that end we prove:

**Theorem 5.1** *Suppose  $\int_0^\infty N(t)^3 dt = K < \infty$ ,  $\xi(0) = 0$ , and  $u$  is a minimal mass blowup solution to (1.1),  $\mu = \pm 1$ . Then  $u(t, x) \in H_x^s(\mathbf{R}^d)$  for  $0 \leq s < 1 + 4/d$  and*

$$\|u(t, x)\|_{L_t^\infty H_x^s((0, \infty) \times \mathbf{R}^d)} \lesssim K^{s+}.$$

Recall from (2.25) that we also have

$$\sum_{J_k} N(J_k) \sim K.$$

This implies  $|\xi(t_1) - \xi(t_2)| \lesssim_d K$  for all  $t_1, t_2 \in (0, \infty)$ . Therefore  $|\xi(t)| \lesssim_d K$  on  $(0, \infty)$ .

**Theorem 5.2** *If theorem 5.1 is true, a minimal mass blowup solution to (1.1),  $\mu = +1$ , with  $N(t) \leq 1$  and*

$$\int_0^\infty N(t)^3 dt = K < \infty$$

*does not exist.*

*Proof:* Recall the compactness modulus function  $C(\eta)$  defined for all  $0 < \eta < \infty$  from (2.12) and (2.13). There exists a function  $\eta(t)$  such that for  $1 < s < 1 + 4/d$ ,

$$\lim_{t \rightarrow \pm\infty} C(\eta(t))N(t) + \eta(t)^{\frac{s-1}{2s}} = 0. \quad (5.2)$$

So for any  $\delta > 0$  there exists  $T$  sufficiently large so that

$$C(\eta(T))N(T) + \eta(T)^{\frac{s-1}{2s}} < \delta.$$

Make a Galilean transformation setting  $\xi(T) = 0$ .

$$\|u(T)\|_{\dot{H}^1(\mathbf{R}^d)} \lesssim \|u_{|\xi| \leq C(\eta(T))N(T)}\|_{\dot{H}^1(\mathbf{R}^d)} + \|u_{|\xi| \geq C(\eta(T))N(T)}\|_{\dot{H}^1(\mathbf{R}^d)} \lesssim C(\eta(T))N(T) + \eta(T)^{\frac{s-1}{2s}}. \quad (5.3)$$

The estimate on  $u_{|\xi| \geq C(\eta(T))N(T)}$  follows from interpolating  $\|u_{|\xi| \geq C(\eta(T))N(T)}\|_{L^2(\mathbf{R}^d)} < \eta(T)^{1/2}$  with

$$\|u(t)\|_{L_t^\infty \dot{H}_x^s((0, \infty) \times \mathbf{R}^d)} \lesssim K^{s+} \quad (5.4)$$

for  $1 < s < 1 + 4/d$ . Before we made the Galilean transformation that set  $\xi(T) = 0$ , we had  $|\xi(t)| \lesssim_d K$  for all  $t \in (0, \infty)$ , so by the triangle inequality and (5.4), after the Galilean transformation,

$$\|u(T)\|_{\dot{H}_x^s(\mathbf{R}^d)} \lesssim K^{s+}. \quad (5.5)$$

This bound is uniform for  $T \in [0, \infty)$ . Also, by the Sobolev embedding theorem,

$$\|u(T)\|_{L_x^{\frac{2(d+2)}{d}}(\mathbf{R}^d)}^{\frac{2(d+2)}{d}} \lesssim_d \|u(T)\|_{\dot{H}_x^{\frac{d}{d+2}}(\mathbf{R}^d)}^{\frac{2(d+2)}{d}} \lesssim_d \|u(T)\|_{\dot{H}^1(\mathbf{R}^d)}^2 \|u(T)\|_{L^2(\mathbf{R}^d)}^{4/d} \lesssim_{m_0, d} \delta^2. \quad (5.6)$$

Using conservation of energy and (1.5), for all  $t \in (0, \infty)$ ,

$$E(u(T)) = E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{d}{2d+4} \int |u(t, x)|^{\frac{2d+4}{d}} dx \lesssim_{m_0, d} \delta^2. \quad (5.7)$$

By (2.12) and conservation of mass,

$$\frac{99m_0^2}{100} < \int_{|x-x(0)| < \frac{1}{N(0)} C(\frac{m_0^2}{100})} |u(0, x)|^2 dx,$$

which by Holder's inequality and conservation of energy,

$$\leq \frac{1}{N(0)^{\frac{2d}{d+2}}} C(\frac{m_0^2}{100})^{\frac{2d}{d+2}} \|u(0)\|_{L_x^{\frac{2(d+2)}{d}}(\mathbf{R}^d)}^2 \leq \frac{1}{N(0)^{\frac{2d}{d+2}}} C(\frac{m_0^2}{100})^{\frac{2d}{d+2}} E(T)^{\frac{d}{d+2}} \lesssim \frac{1}{N(0)^{\frac{2d}{d+2}}} \cdot C(\frac{m_0^2}{100})^{\frac{2d}{d+2}} \delta^{\frac{d}{d+2}}.$$

For  $\delta > 0$  very small this is a contradiction. Therefore theorem 5.2 has been proved, assuming theorem 5.1 is true.  $\square$

**Remark:** We could also apply this argument to the case  $\mu = -1$ ,  $\|u_0\|_{L^2(\mathbf{R}^d)}$  is less than the mass of the ground state. We will not bother to do that here.

*Proof of theorem 5.1:* We will rely on two intermediate lemmas to prove theorem 5.2. As usual we will partition  $(0, \infty)$  into subintervals  $J_k$  with  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} = \epsilon$ .

**Lemma 5.3** *For any  $1/2 \leq \lambda < 1 + 4/d$  and  $\lambda \leq 1/2 + \sigma$ , if*

$$\sup_{J_k} \|u_{>M}\|_{S_*^0(J_k \times \mathbf{R}^d)} \lesssim_{m_0, d, \sigma} \frac{K^\sigma}{M^\sigma}, \quad (5.8)$$

*then*

$$\|P_{|\xi| \geq N}(|u|^{4/d} u)\|_{S^0((0, \infty) \times \mathbf{R}^d)} \lesssim_{m_0, d, \lambda} \frac{K^\lambda}{M^\lambda}. \quad (5.9)$$

*Proof:* We have already proved that for any compact interval  $J$ , when  $N \leq K$ ,

$$\|P_{|\xi-\xi(t)|>N}u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \lesssim \frac{K^{1/2}}{N^{1/2}}. \quad (5.10)$$

Let  $C$  be a large, fixed constant such that  $|\xi(t_1) - \xi(t_2)| \leq \frac{C}{1000}K$ . When  $N \leq CK$ , take  $J_n = [0, T_n]$ . By theorem 3.1, with implied constant independent of  $T_n$ ,

$$\|u_{|\xi-\xi(t)| \geq N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0, T_n] \times \mathbf{R}^d)} \lesssim_{m_0, d} \frac{K^{1/2}}{N^{1/2}}. \quad (5.11)$$

Taking  $T_n \rightarrow \infty$ , we have

$$\|u_{|\xi-\xi(t)| \geq N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}((0, \infty) \times \mathbf{R}^d)} \lesssim_{m_0, d} \frac{K^{1/2}}{N^{1/2}}. \quad (5.12)$$

In fact, for any  $\lambda \geq 1/2$ , when  $N \leq CK$ ,

$$\|u_{|\xi-\xi(t)| \geq N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}((0, \infty) \times \mathbf{R}^d)} \lesssim_{m_0, d, \lambda} \frac{K^\lambda}{N^\lambda}. \quad (5.13)$$

Interpolating this with conservation of mass,

$$\|u_{|\xi-\xi(t)| \geq N}\|_{S^0((0, \infty) \times \mathbf{R}^d)} \lesssim_{m_0, d, \lambda} \frac{K^\lambda}{N^\lambda}. \quad (5.14)$$

Now we can use

**Lemma 5.4** *Let  $u$  be a solution to (1.1) which is almost periodic modulo scaling on its maximal lifespan  $I$ ,  $u$  blows up forward in time. Then for all  $t \in I$ ,*

$$u(t) = \lim_{T \nearrow \sup I} i \int_t^T e^{i(t-\tau)\Delta} F(u(\tau)) d\tau, \quad (5.15)$$

as a weak limit in  $L_x^2$ .

*Proof:* See section 6 of [33].  $\square$

For  $N \geq CK$ ,

$$\|P_{|\xi-\xi(t)|>N}u\|_{S^0((0, \infty) \times \mathbf{R}^d)} \lesssim_d \|P_{|\xi| \geq \frac{N}{2}}(|u|^{4/d}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0, \infty) \times \mathbf{R}^d)}. \quad (5.16)$$

$$\begin{aligned}
& \lesssim \|P_{|\xi| \geq \frac{N}{2}}(|u_{|\xi-\xi(t)| \leq \eta N}|^{4/d} u_{|\xi-\xi(t)| \leq \eta N})\|_{L_t^2 L_x^{\frac{2d}{d+2}}((0,\infty) \times \mathbf{R}^d)} \\
& + \|(1 - \chi(t))u\|_{L_t^\infty L_x^2((0,\infty) \times \mathbf{R}^d)}^{4/d} \|u_{|\xi-\xi(t)| \geq \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}((0,\infty) \times \mathbf{R}^d)} \\
& + \|u_{|\xi-\xi(t)| \geq C_0 N(t)}\|_{L_t^\infty L_x^2((0,\infty) \times \mathbf{R}^d)}^{4/d} \|u_{|\xi-\xi(t)| \geq \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}((0,\infty) \times \mathbf{R}^d)} \\
& + \|(u_{|\xi-\xi(t)| \geq \eta N})|\chi(t)u_{|\xi-\xi(t)| \leq C_0 N(t)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}((0,\infty) \times \mathbf{R}^d)}.
\end{aligned} \tag{5.17}$$

Therefore, for any  $0 < s < 1 + 4/d$ ,

$$(5.17) \lesssim_{m_0, d, s} \sum_{M \leq \eta N} \left(\frac{M}{N}\right)^s \|u_{|\xi-\xi(t)| \geq M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}((0,\infty) \times \mathbf{R}^d)} + \delta(C_0) \|P_{|\xi-\xi(t)| \geq \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}((0,\infty) \times \mathbf{R}^d)} \tag{5.18}$$

$$\begin{cases} +C_0^{3/2}(\frac{K}{\eta N})^{1/2}(\sup_{J_k} \|u_{|\xi-\xi(t)| \geq \frac{\eta N}{2}}\|_{S_*^0(J_k \times \mathbf{R}^d)}), & \text{if } d = 3; \\ +C_0^{4-6/d}(\frac{K}{\eta N})^{2/d}(\sup_{J_k} \|u_{|\xi-\xi(t)| \geq \frac{\eta N}{2}}\|_{S_*^0(J_k \times \mathbf{R}^d)})^{4/d} \|u_{|\xi-\xi(t)| \geq \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}((0,\infty) \times \mathbf{R}^d)}^{1-4/d}, & \text{if } d \geq 4. \end{cases} \tag{5.19}$$

By induction,

$$\begin{aligned}
& \|P_{|\xi| \geq \frac{N}{2}}(|u|^{4/d} u)\|_{S^0((0,\infty) \times \mathbf{R}^d)} \leq \sum_{M \leq \eta N} C_2(m_0, d, s) C_3(m_0, d, \lambda) \left(\frac{K}{M}\right)^\lambda \eta^{s-\lambda} \\
& + \delta(C_0) C_2(m_0, d, s) C_3(m_0, d, \lambda) \left(\frac{K}{\eta N}\right)^\lambda \\
& + \begin{cases} C_2(m_0, d, s) \left(\frac{K^{1/2}}{N^{1/2}}\right) \left(\frac{K^\sigma}{N^\sigma}\right) \frac{C_0^{3/2}}{\eta^{1/2+\sigma}}, & \text{if } d = 3; \\ C_2(m_0, d, s) C_3(m_0, d, \lambda)^{1-4/d} \left(\frac{K}{\eta N}\right)^{2/d} \left(\frac{K}{\eta N}\right)^{4\sigma/d} \left(\frac{K}{\eta N}\right)^{(1-4/d)\lambda} C_0^{4-6/d}, & \text{if } d \geq 4. \end{cases}
\end{aligned}$$

If  $\lambda < 1 + 4/d$  we can find  $s$  such that  $\lambda < s < 1 + 4/d$ . Take  $s = \frac{\lambda+1+4/d}{2}$ . Choose  $\eta$  sufficiently small so that  $\eta^{s-\lambda} C_2$  is very small. Then take  $C_0(d, s, \eta, \lambda)$  sufficiently large so that  $\frac{\delta(C_0)}{\eta^\lambda} C_2$  is very small. Finally, if  $d = 3$  choose  $C_3$  sufficiently large so that

$$\frac{C_2 C_0^{3/2}}{\eta^\lambda} << C_3,$$

and if  $d \geq 4$  choose  $C_3$  sufficiently large so that

$$\frac{C_2 C_0^{4-6/d}}{\eta^\lambda} << C_3^{4/d}.$$

This closes the induction and completes the proof.  $\square$

**Remark:** We assume  $K \ll \eta N$ , otherwise we just use the results of §3.

Now suppose  $I$  is some interval  $[a, b]$ .

**Lemma 5.5** *If  $u$  is a solution to (1.1),*

$$\|P_{>N}u(a)\|_{L_x^2(\mathbf{R}^d)} \lesssim_{m_0,d,\lambda} \frac{K^\lambda}{N^\lambda}, \quad (5.20)$$

with  $\lambda < 1 + 4/d$ , and

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)} \leq \delta \quad (5.21)$$

for some  $\delta(m_0, d, \lambda) > 0$  sufficiently small, then

$$\|P_{>N}u\|_{S_*^0(I \times \mathbf{R}^d)} \lesssim_{m_0,d,\lambda} \frac{K^\lambda}{N^\lambda}. \quad (5.22)$$

*Proof:* By Duhamel's formula and (2.4),

$$\|P_{>N}u\|_{S_*^0(I \times \mathbf{R}^d)} \equiv \|P_{>N}u(a)\|_{L_x^2(\mathbf{R}^d)} + \|P_{>N}(|u|^{4/d}u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I \times \mathbf{R}^d)}. \quad (5.23)$$

Since

$$\|u\|_{S_*^0(J_k \times \mathbf{R}^d)} \lesssim_{m_0,d} 1 + \delta^{1+4/d} \lesssim_{m_0,d} 1,$$

our lemma is true for  $N \leq CK$ . By Bernstein's inequality and corollary 2.10,

$$\begin{aligned} \|P_{>N}(|u|^{4/d}u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I \times \mathbf{R}^d)} &\lesssim_d \|P_{>N}(|u_{\leq N}|^{4/d}u_{\leq N})\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} + \| |u_{>N}|^{4/d} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\ &\lesssim_{m_0,d,s} \sum_{M \leq N} \left(\frac{M}{N}\right)^s \|P_{>M}u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)}^{4/d} \\ &\lesssim_{m_0,d,s} \sum_{M \leq N} \left(\frac{M}{N}\right)^s \|P_{>M}u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)} \delta^{4/d}. \end{aligned} \quad (5.24)$$

Then apply the method of continuity. Recursively define a sequence of functions,

$$\begin{aligned} u_0 &= e^{it\Delta}u(0), \\ u_{n+1} &= e^{it\Delta}u_0 - i \int_0^t e^{i(t-\tau)\Delta} |u_n(\tau)|^{4/d} u_n(\tau) d\tau. \end{aligned} \quad (5.25)$$



By (5.20) and Strichartz estimates,

$$\|P_{>N}e^{it\Delta}u(0)\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)} \lesssim_d \frac{K^\lambda}{N^\lambda}.$$

Let  $s = \frac{\lambda+1+4/d}{2}$ ,

$$\sup_N \left(\frac{N}{K}\right)^\lambda \|P_{>N}u_{n+1}\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)} \lesssim_{m_0,d,\lambda} 1 + \delta^{4/d} \left(\sup_N \left(\frac{N}{K}\right)^\lambda \|P_{>N}u_n\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)}\right).$$

By continuity, for  $\delta(m_0, d, \lambda) > 0$  sufficiently small

$$\|u_{>N}\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)} \lesssim_{m_0,d,\lambda} \frac{K^\lambda}{N^\lambda}.$$

By the same argument we also have

$$\|P_{>N}(|u|^{4/d}u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I \times \mathbf{R}^d)} \lesssim_{m_0,d,\lambda} \frac{K^\lambda}{N^\lambda}. \quad (5.26)$$

Therefore,

$$\|u_{>N}\|_{S_*^0(I \times \mathbf{R}^d)} \lesssim_{m_0,d,\lambda} \frac{K^\lambda}{N^\lambda}. \quad (5.27)$$

Again, since  $|\xi| \sim |\xi - \xi(t)|$  when  $N \geq CK$ , this proves

$$\|u_{|\xi - \xi(a)| > N}\|_{S_*^0(I \times \mathbf{R}^d)} \lesssim \frac{K^\lambda}{N^\lambda}. \quad (5.28)$$

□

**Corollary 5.6** *If  $u$  is a solution to (1.1),  $\lambda < 1 + 4/d$ ,*

$$\|P_{>N}u\|_{L_t^\infty L_x^2((0,\infty) \times \mathbf{R}^d)} \lesssim \frac{K^\lambda}{N^\lambda}, \quad (5.29)$$

and

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} = \epsilon, \quad (5.30)$$

then

$$\|u_{>N}\|_{S_*^0(J_k \times \mathbf{R}^d)} \lesssim \frac{K^\lambda}{N^\lambda}. \quad (5.31)$$

*Proof:* Partition each subinterval  $J_k$  with  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k \times \mathbf{R}^d)} = \epsilon$  into a finite number of subintervals  $I_i$  with  $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I_i \times \mathbf{R}^d)} = \delta$ . Combining (5.29) and lemma 5.5,

$$\|P_{>N}u\|_{S_*^0(J_k \times \mathbf{R}^d)} \lesssim \frac{K^\lambda}{N^\lambda}. \quad (5.32)$$

Now we are ready to prove theorem 5.1.

*Proof of theorem 5.1:* This is proved by induction. Take  $N \geq CK$ . Lemma 5.3 implies that since  $\|u\|_{S_*^0(J_k \times \mathbf{R}^d)} \lesssim 1$ ,

$$\|u_{|\xi - \xi(t)| > N}\|_{S^0((0,\infty) \times \mathbf{R}^d)} \lesssim_{m_0,d} \frac{K^{1/2}}{N^{1/2}}.$$

By corollary 5.6 this implies

$$\|P_{|\xi - \xi(a_k)| > N}u\|_{S_*^0(J_k \times \mathbf{R}^d)} \lesssim \frac{K^{1/2}}{N^{1/2}}.$$

Applying lemma 5.3 again we have

$$\|u_{|\xi - \xi(t)| > N}\|_{S^0((0,\infty) \times \mathbf{R}^d)} \lesssim \frac{K}{N}.$$

Iterating at most four more times, theorem 5.1 is proved.  $\square$

We have excluded the second minimal mass blowup scenario. This concludes the proof of theorem 1.6.  $\square$

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